

## ANOTHER SIMPLE PROOF OF THE VALIDITY OF NAGAMUCHI AND IBARAKI'S MIN-CUT ALGORITHM AND QUEYRANNE'S EXTENSION TO SYMMETRIC SUBMODULAR FUNCTION MINIMIZATION

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*Abstract* M. Stoer and F. Wagner and independently A. Frank have found a simple proof of the validity of Nagamochi and Ibaraki's min-cut algorithm. This note points out some nice property of the behavior of Nagamochi and Ibaraki's min-cut algorithm, which also gives another simple proof of the validity of their algorithm. The proof relies only on the symmetric submodularity of the cut function. Hence, it also gives another simple proof of the validity of Queyranne's extension of Nagamochi and Ibaraki's algorithm to symmetric submodular function minimization.

### 1. Introduction

M. Stoer and F. Wagner [7] have found a simple and elegant proof of the validity of Nagamochi and Ibaraki's min-cut algorithm [4] for undirected networks, which substantially simplifies the original proof in [4]. Also, A. Frank [1] has independently obtained another simple proof, where the proof is given for the (unweighted) edge-connectivity algorithm but its adaptation for the weighted case is straightforward as remarked in [1]. Moreover, based on the results of [1] and [7], M. Queyranne [6] extended Nagamochi and Ibaraki's algorithm to the minimization of symmetric submodular functions; the cut function for an undirected network is a typical example of a symmetric submodular function in the sense of [3].

This note points out some nice property of the behavior of Nagamochi and Ibaraki's min-cut algorithm, which also gives another simple proof of the validity of their algorithm. The proof relies only on the symmetric submodularity of the cut function. Hence, it also gives another simple proof of the validity of Queyranne's extension [6] of Nagamochi and Ibaraki's algorithm to symmetric submodular function minimization.

### 2. A Simple Proof

Let  $G = (V, E)$  be an undirected graph with a positive weight (or capacity) function  $w$  on the edge set  $E$ . Suppose that using the procedure CAPFOREST (a breadth-first method) proposed by Nagamochi and Ibaraki [4], we have chosen (or visited) the vertices  $v_1, v_2, \dots, v_n$  ( $n = |V|$ ) in this order. Frank [1] called this sequence a legal ordering (the term was first used for multigraphs in [2]). Define  $U_k = \{v_1, v_2, \dots, v_k\}$  for  $k = 0, 1, \dots, n$  and let  $G_k$  be the graph obtained by removing from  $G$  the edges connecting between vertices in  $V - U_{k-1}$  for  $k = 1, 2, \dots, n$ . Note that  $G_k$  is the spanning subgraph of  $G$  having the edges that have been traversed by CAPFOREST when  $v_k$  is to be determined. Denote by  $w_k(C)$  the weight (or capacity) of a cut  $C$  in graph  $G_k$  for  $k = 1, 2, \dots, n$ . (Here, note that a cut  $C$  is a vertex subset and that the weight (or capacity) of cut  $C$  is the sum of the weights of the edges

between  $C$  and  $V - C$ .) Recall that for each  $k = 1, 2, \dots, n$ ,  $v_k$  is a vertex  $u \in V - U_{k-1}$  that maximizes  $w_k(\{u\})$ . We will write  $w_k(u)$  instead of  $w_k(\{u\})$  for  $u \in V$  in the following.

Now, we show the following lemma. The validity of Nagamochi and Ibaraki's min-cut algorithm easily follows from this lemma.

**Lemma 2.1:** *For any  $k = 1, 2, \dots, n - 1$  and any  $u \in V - U_k$ ,  $\{u\}$  is a  $u-v_k$  min-cut in  $G_k$ .*

(Proof) We show this lemma by induction. For  $k = 1$  it trivially holds. So, suppose that it holds for  $k = l$  with  $1 \leq l < n - 1$ . Consider any  $u \in V - U_{l+1}$  and any  $u-v_{l+1}$  cut  $C$ . If  $C$  is a  $u-v_l$  cut, then by the induction hypothesis ( $w_l(C) \geq w_l(u)$ ) and by the definitions of  $G_l$  and  $G_{l+1}$  we have

$$w_{l+1}(C) - w_{l+1}(u) \geq w_l(C) - w_l(u) \geq 0. \tag{2.1}$$

Also, if  $C$  is a  $v_l-v_{l+1}$  cut, then by the induction hypothesis ( $w_l(C) \geq w_l(v_{l+1})$ ) and by the definition of  $v_{l+1}$  we have

$$w_{l+1}(C) \geq w_{l+1}(v_{l+1}) \geq w_{l+1}(u). \tag{2.2}$$

Hence, the lemma holds for  $k = l + 1$ . This completes the proof. □

This lemma also easily follows from the result of Nagamochi and Ibaraki [4] (but recall that our objective is to give a simple proof of the validity of their algorithm). It also follows from a lemma of Stoer and Wagner [7] and from a lemma of Frank [1]. Very recently, M. Queyranne [6] has found a (surprising) strongly polynomial combinatorial algorithm for minimizing symmetric submodular functions by generalizing the arguments in Stoer and Wagner [7]. The above lemma is almost the same as (but slightly different from) Lemma 1 of Queyranne [6] when specialized to cut functions of undirected graphs but the proofs are quite different.

The above lemma can also be slightly strengthened as follows.

**Lemma 2.2:** *For any  $k = 1, 2, \dots, n - 1$  and any distinct  $u, v \in V - U_{k-1}$ , if  $w_k(u) \leq w_k(v)$ , then  $\{u\}$  is a  $u-v$  min-cut in  $G_k$ .*

(Proof) Just replace  $v_{l+1}$  by  $v$  in the above proof of Lemma 1. □

These lemmas, especially Lemma 2.2, show more clearly the behavior of Nagamochi and Ibaraki's algorithm.

The above proofs also work for a symmetric submodular system  $(V, f)$  ([3]) by defining

$$w_k(C) = f(C) - (1/2)\{f(C \cap \overline{U_{k-1}}) + f(\overline{C} \cap \overline{U_{k-1}}) - f(\overline{U_{k-1}})\} \tag{2.3}$$

for each  $k = 1, 2, \dots, n - 1$  and  $C \subseteq V$ , where  $\overline{X}$  denotes the complement of  $X$  in  $V$  and  $U_k$  ( $k = 0, 1, \dots, n$ ) are defined by the same procedure as described above by using this  $w_k$  for each  $k = 1, 2, \dots, n$ , i.e.,  $v_k \in V - U_{k-1}$  attains the maximum value of  $w_k(\{u\})$  ( $u \in V - U_{k-1}$ ). The procedure is exactly the same as Queyranne's [6]. The present proof looks simpler than that in [6]. Note that when  $f$  is the cut function of a given weighted graph, the present  $w_k$  in (2.3) is exactly the same as that defined for the graph.

The above lemmas are useful for finding  $s-t$  min-cuts in  $G$ . Suppose that for some  $k < n - 1$  there exists a vertex  $u \in V - U_k$  such that  $u$  is not adjacent in  $G$  to any other vertices in  $V - U_k$ . Then we can conclude from the above lemmas that  $\{u\}$  is a  $u-v_k$  min-cut in  $G$  and we can finish CAPFOREST. Also, instead of truncating the procedure CAPFOREST we may continue the procedure till the end to get more than one  $s-t$  pair and then merge each found  $s-t$  pair at the end of the procedure. This may be more effective in practical implementation of the algorithm. An efficient implementation technique was also given in [5].

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