

## DISCRETE TIME MULTI-CLASS FEEDBACK QUEUE WITH VACATIONS AND CLOSE TIME UNDER RANDOM ORDER OF SERVICE DISCIPLINE

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*Abstract* We study a discrete-time single server priority queueing model with vacations under random order of service discipline within each class. This model captures the behavior of the head-of-line request queues in large input-buffered ATM switches. The server takes vacations when the queue has been empty for a random number of slots. We presume a message consists of a geometrically distributed number of cells. To represent this aspect, we assume that once a message gets in turn for service, it is served for a constant time which corresponds to one-cell-time and it rejoins the queue after service with a given probability. We derive the joint probability distribution of the queue length and waiting time through probability generating function approach. Mean waiting times are obtained and their numerical results are shown.

### 1. Introduction

In recent years, telecommunication traffic volume is rapidly increasing as various multimedia applications are being developed and the coverage of communication networks becomes widespread to accommodate more users. In order to integrate these types of traffic satisfying the requested QoS in a common network, ATM technique has been developed and the performance evaluation of ATM switches is strongly desired. In the performance analysis of the input-buffered ATM switches, the introduction of the concept of virtual queues is important [7]. The virtual queue consists of messages at the head of each input-buffer competing for the same output-port. We regard the virtual queue and the output-port as a system, and estimate the performance of the system through queueing theory.

Queueing systems have been extensively studied and applied in the performance evaluation of computer systems and communication networks, and a good compilation of results on the M/G/1 queueing system and its variants can be found in [10]. M/G/1 queues with vacations [3] serve as simplified models in analyzing cyclic service network protocols. By vacation, we mean that the server becomes unavailable for occasional intervals of time. Variants of queueing model with priorities [5] can be used to mathematically describe networks that support a variety of traffic classes with different service requirements. In the vast literature on single server queues, many of them treat continuous time systems under FIFO discipline. In ATM networks, a message is divided into cells, blocks of a constant length, before transmission and cells are switched in a constant time. Thus an ATM switch is often modeled as a discrete time queueing system. Furthermore in these studies on the performance of ATM switches, the order of service is not FIFO in many cases. Especially, random order of service discipline (ROS) is significant in the context of performance analysis of input-output-buffered ATM switches [1]. Under ROS, the next message for service is selected at random from the messages of the highest priority class waiting in the queue.

Takagi [11] widely studies discrete time systems including those under ROS. However, as Laevens and Bruneel [8] point out, the result for ROS systems in [11] is rather complicated and seems difficult to be applied to performance analysis of various ROS systems as it is.

In this paper we use the technique introduced in [6, 7, 8]. In [6] the waiting time under ROS discipline for messages consisting of a single ATM-cell is analyzed, and the result is extended to cover the case of variable-length messages in [8] and the case where a single ATM-cell may feedback in [7]. We analyze a discrete time queueing system with vacations each of which is initiated after the expiration of close-time without message arrivals. We study a system where variable-length-messages belong to one of two priority classes. The number of feedbacks corresponds to the length of messages. We assume preemptive service where a message of normal priority class in service is put aside upon arrival of a message of high priority class. This queueing model is an extension of previously analyzed models in [6, 7, 8] and provides more flexibility in performance studies.

In ATM networks, all information including voice, video, and data, is conveyed in an integrated fashion. The quality-of-service (QoS) required by different applications is highly varied. We classify cells in two types, cells including real-time-data and those including non-real-time-data. Since long delay is fatal for real-time-data, a group of cells of this type is modeled as high-priority-message, while a group of cells conveying non-real-time-data is modeled as normal-priority-message. Service priorities are not explicitly indicated in the header of cells, however are associated implicitly with a VP or VC [2]. A call is the connection request to transfer information stream of multimedia traffic. When a source makes a call to a destination, the source has to first require VCs of ATM networks. As a new call gets VCs, following two events happen. 1) Call admission control receives the request of new VCs from a new call and checks the status of the network whether new VCs are able to be established under QoS required by the call and 2) VCs are established. It is necessary to limit the number of accepted calls in ATM networks to satisfy QoS requested by calls. Call admission control decides properly whether a new VC is accepted based upon its anticipated traffic characteristics, its QoS requirements and the current network load [12]. In ATM networks, keeping VC connection while no cell is transmitted wastes resources, especially buffers, and this should be avoided by putting the threshold concerning the length of inactive periods. Once the threshold has been expired, the connection is automatically closed to increase the utilization of network resources. We can model this mechanism by using a queueing model with close-time, which corresponds to the threshold for disconnection. The time from the instance when a call requests VCs to the instance the call actually gets VCs is considered as a period of service suspension and modeled as the remaining time of vacation in our system. Even under low utilization of network resources, new VCs are only provided after call admission control admits their connection requests and there is some time-lag between occurrences of requests and their set-ups. Under high utilization of network resources, the meaning of vacation is more significant. Under this situation, new VCs are not allowed upon a request due to network congestion. We show some examples of these situations in the following. 1) Calls from various applications occupy a lot of VCs. In this case a new call has to wait until some applications release their VCs. 2) The network supports ABR(Available Bit Rate) service. In ABR service, the network provides rate feedback to the sender asking it to slow down when congestion occurs. In this case, the rest of the available bandwidth after allocated to preceding VCs might be highly utilized. When a new call is requested, the network first puts back pressure to ABR traffic sources in order to reduce their transmission rates. It takes some time for new VCs to be set up after the utilization gets lower.

The main goal of this paper is to present a new method to estimate the performance of ATM switches using ROS queueing system with vacations, feedback, priorities, and close-time, which generalizes the queueing system by Laevens and Bruneel [7, 8, 6]. The approach taken here is by means of the probability generating function (PGF) technique. It should be noted that our approach is based upon that of Laevens and Bruneel and enables one to treat complicated discrete queueing systems under ROS discipline easily.

This paper is organized as follows. We describe the model and introduce notation in section 2. Section 3 derives the queue size distribution at various observing points through generating function approach. Using the results in section 3, section 4 describes the derivation of the waiting time distribution for both high- and normal-priority messages. Section 5 presents the numerical results for the mean waiting time of both classes of messages under various scenarios.

## 2. The Model

We consider a discrete-time queueing model. Time is thus divided into consecutive intervals of a fixed length, called slots. There is one single server and the system has an infinite buffer capacity, so that no message is lost due to buffer overflow. There are two kinds of messages arriving at the system. They are indexed as  $M_H$  and  $M_N$ .  $M_H$  messages have priority over  $M_N$  messages. That is,  $M_N$  messages are served iff there is no  $M_H$  message in the queue. The order of the service for messages of the same kind is random(ROS). That is the next message for service is selected at random from those waiting in the queue. The arrival process is general and independent, i.e., the number of new arrivals during each slot is assumed to be independent and identically distributed (i.i.d.). We denote by  $a_p^{(k)}$  the number of new arrivals of  $M_p$  ( $p = H, N$ ) messages during slot  $k$ . Then, all  $a_p^{(k)}$  are i.i.d. with common probability generating function  $A_p(x_p) \equiv E[x_p^{a_p^{(k)}}]$  ( $p = H, N$ ).

Every message receives service of one slot once it reaches the server. If afterwards it requires more service, it has to rejoin the queue and wait once more until it wins the right to be served. Messages rejoining the queue form an internal arrival process, while messages arriving to the queue from outside the queue form an external arrival process. We assume the number of slots for service which is required by a message is geometrically distributed with parameter  $\alpha_H$  ( $\alpha_N$ ) for  $M_H$  ( $M_N$ ) messages.

When the server finishes serving a message and finds the system empty, it undergoes close-time for a some number of slots. We introduce the probability generating function(PGF)  $C(z)$  for the number of slots in a close-time. If there is no message in the system when a close-time ends, the server takes a vacation, otherwise it begins service for messages upon their arrivals. Note that a close-time not always lasts as long as 'predicted' by the distribution characterized by  $C(z)$ . In other words, the length of an actual close time may be distributed differently from  $C(z)$ . We assume that the length of a vacation is i.i.d. and define  $V(z)$  as the PGF for the number of slots in a vacation. At the end of each vacation, it checks again if there are messages waiting in the queue. When it finds any, it begins to serve. Otherwise it takes another vacation. Later on this procedure is repeated.

The equilibrium condition for  $M_H$  messages is given by

$$\frac{\lambda_H}{1 - \alpha_H} < 1,$$

and that for  $M_N$  messages is given by

$$\frac{\lambda_H}{1 - \alpha_H} + \frac{\lambda_N}{1 - \alpha_N} < 1,$$

where  $\lambda_H$  is the mean number of  $M_H$  messages arriving in a slot,  $\frac{d}{dx_H} A_H(x_H) |_{x_H=1}$  and  $\lambda_N$  is the mean number of  $M_N$  messages arriving in a slot,  $\frac{d}{dx_N} A_N(x_N) |_{x_N=1}$ .

### 2.1. Classification of the System States

In this section we introduce periods and cycles to make clear the behavior of the system. A set of designated slots successively located on the time axis is called period. We define cycle as a set of slots or periods to capture the cyclic behavior of the system.

We define C-period as a period consisting of slots in which the server undergoes close-time. Similarly we define V-period as a period the server is taking vacations, and B-period as a period the server is devoted to service. We define Basic-cycle as a successive sequence of C-period followed by V-period if any and B-period. At the beginning and the end of Basic-cycle, the number of messages in the system is zero. Whether V-period appears in Basic-cycle depends on whether any  $M_H$  or  $M_N$  messages arrive during C-period. On the contrary, exactly one C-period and B-period appear in each Basic-cycle. See Figure 1.

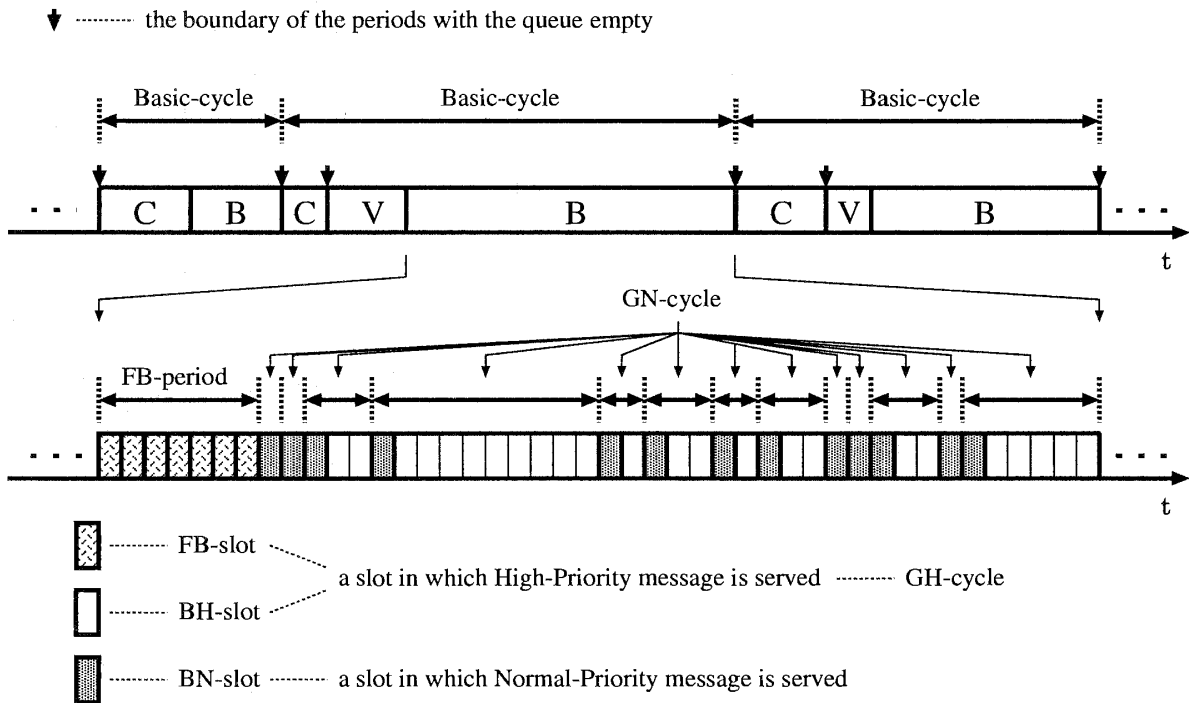


Figure 1: Structure of Basic-cycle

We classify slots in B-period into three types, FB-, BH- and BN- slots. FB-slot is a slot in which the server is devoted to the service for either one of  $M_H$  messages waiting in the queue at the end of C-period or V-period, or one of those which arrive to the system in FB-slots. We define FB-period as a successive sequence of FB-slots. If C-period or V-period is not ended by the arrival of  $M_H$  messages but that of  $M_N$  messages, FB-period does not appear in Basic-cycle. In B-period, slots aside from FB-slots are classified into BH-slot or BN-slot, if any. We define BH-slot as a slot in which the server serves  $M_H$  message and

BN-slot as a slot in which the server serves  $M_N$  message. Since there is no  $M_H$  message in the queue at the end of FB-period, it is followed by BN-slot unless no message is in the queue. BN-slot also appears after C-period (V-period) not followed by FB-period. If there are any  $M_H$  messages arriving to the system during this BN-slot, the next slot must be BH-slot. A sequence of BH-slots continues until all  $M_H$  messages in the queue are served. This means no  $M_N$  message is served until then. At the end of the successive BH-slots, unless no  $M_N$  message is in the queue, another BN-slot appears and the process described above is repeated.

We define GN-cycle as a successive sequence of BN-slot followed by BH-slots if any. In this cycle exactly one  $M_N$  message and  $M_H$  messages if any are served. We define GH-cycle as a slot in which one  $M_H$  message is served. This means each FB-slot and BH-slot corresponds to this cycle.

To focus on the behavior of the tagged message, we define TH-cycle and TN-cycle. The first TH-cycle begins at the beginning of the first GH-cycle after the arrival of the tagged  $M_H$  message, and ends at the end of GH-cycle in which the tagged message is served for the first time. If the tagged  $M_H$  message requires more slots for service, we define another TH-cycle as a successive sequence of GH-cycles which begins at the end of the last TH-cycle, and ends at the end of the next GH-cycle in which the tagged  $M_H$  message is served. This procedure is repeated until the tagged  $M_H$  message leaves the system. See Figure 2. In this figure, the tagged  $M_H$  message requires three slots for service before it leaves the system. Similarly, we define TN-cycle. See Figure 3. In this figure, the tagged  $M_N$  message requires two slots for service before it leaves the system.

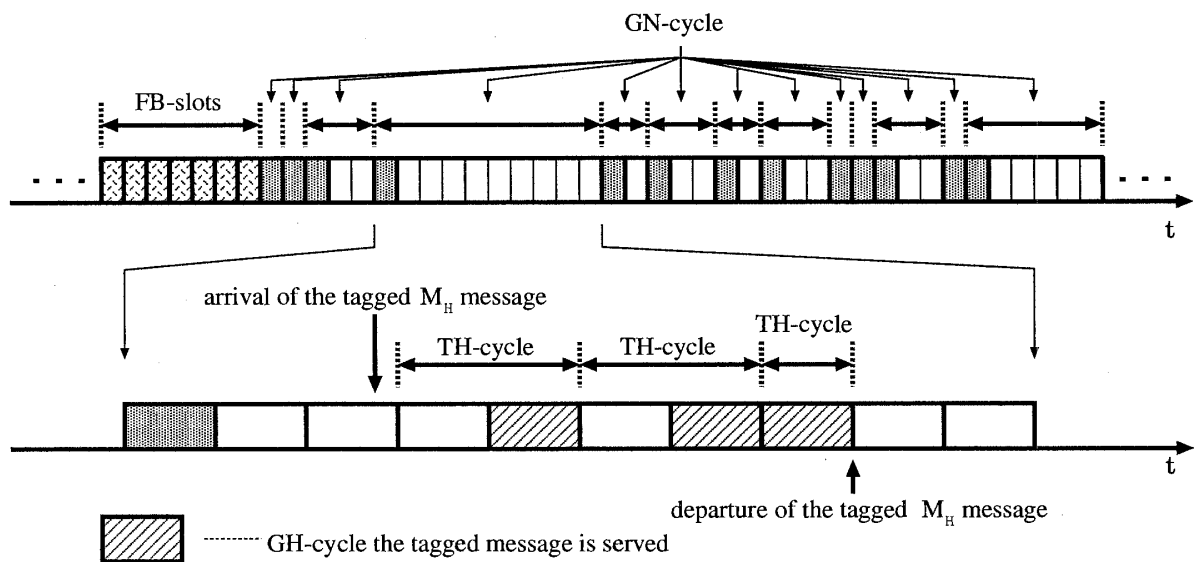


Figure 2: Structure of TH-cycle

### 3. Queue Size Distribution

In this section, we derive the probability generating function (PGF) for the numbers of  $M_H$  and  $M_N$  messages at various observing points.

We define  $\Pi^{FB}(x_H)$  as the PGF for the number of  $M_H$  messages at the departure point of a single cell in FB-periods,  $\Pi^{GN}(x_N)$  as the PGF for the number of  $M_N$  messages at the end of GN-cycles and  $\Pi^{BH}(x_H)$  as the PGF for the number of  $M_H$  messages at the end

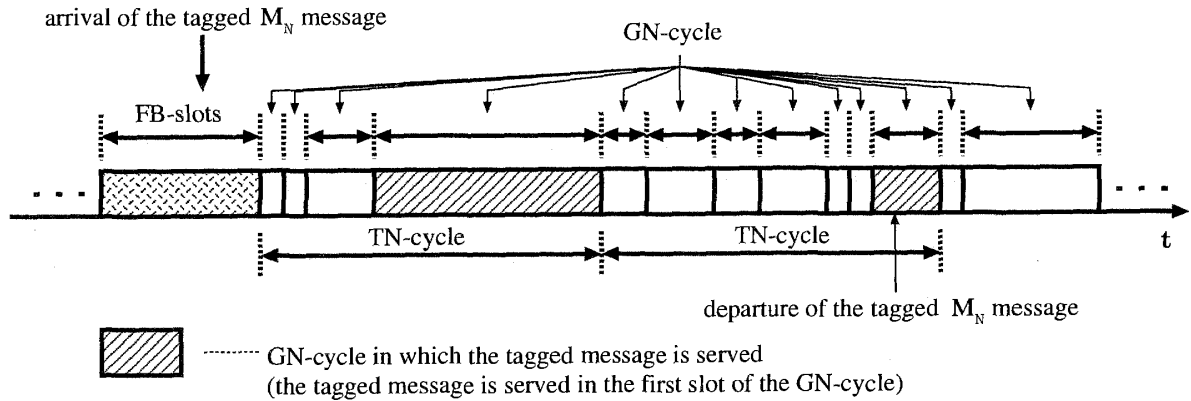


Figure 3: Structure of TN-cycle

of BH-slots. See Figure 4. These PGFs are derived by using the method of the *embedded*

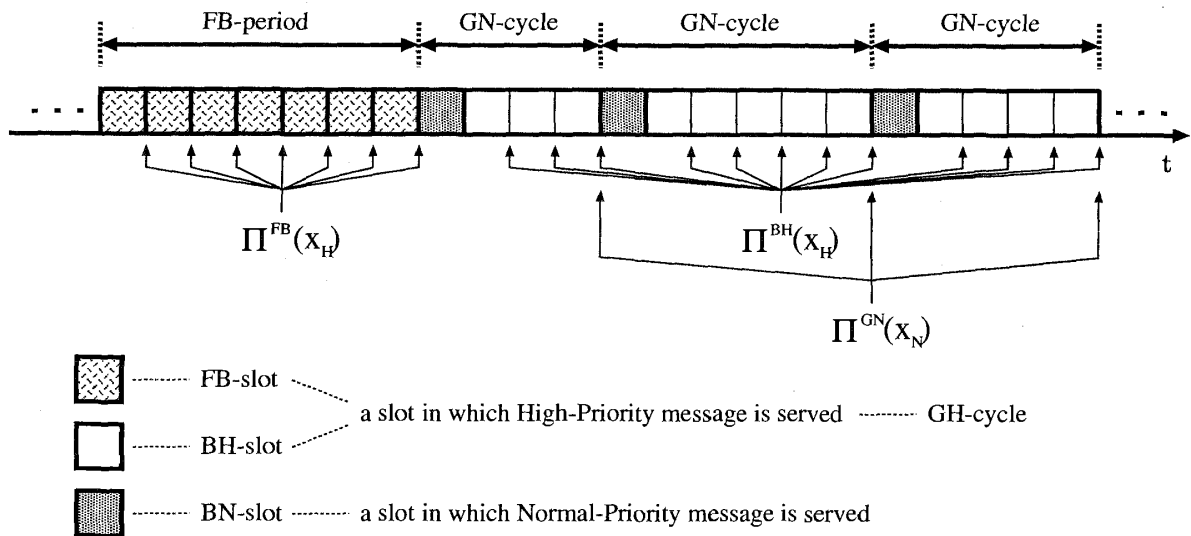


Figure 4: Observing points for each PGF

Markov chain (sec. 1.1 in Takagi[10]).

For the ease of convenience in the following analysis, we define  $B_H(x_H)$ ,  $B_N(x_N)$ ,  $CFB(x_H, x_N)$  and  $VFB(x_H, x_N)$ . By  $B_H(x_H)$  we express the PGF for the number of internally arriving  $M_H$  messages in one GH-cycle as well as in one FB-slot. We also define  $B_N(x_N)$  as the PGF for the number of internally arriving  $M_N$  messages in one GN-cycle and  $CFB(x_H, x_N)$  as the joint PGF for the number of  $M_H$  and  $M_N$  messages at the beginning of FB-period which follows C-period. Similarly to  $CFB(x_H, x_N)$ ,  $VFB(x_H, x_N)$  represents the joint PGF for the number of  $M_H$  and  $M_N$  messages at the beginning of FB-period which

follows V-period. These are given by

$$\begin{aligned}
 B_H(x_H) &= x_H \alpha_H + (1 - \alpha_H) \\
 B_N(x_N) &= x_N \alpha_N + (1 - \alpha_N) \\
 \text{CFB}(x_H, x_N) &= \frac{A_H(x_H) - A_H(0)}{1 - A_H(0)} A_N(x_N) \\
 \text{VFB}(x_H, x_N) &= \frac{V(A_H(x_H)A_N(x_N)) - V(A_H(0)A_N(x_N))}{1 - V(A_H(0))}.
 \end{aligned}$$

Next we derive  $\text{GN}(z)$ , the PGF for the number of slots in one GN-cycle. We define  $\Theta_{H,1}(z)$  as the PGF for the number of slots the server spends before it serves all  $M_H$  messages in the queue on the condition that one  $M_H$  message exists in the system at the beginning of a slot. The PGF for the number of externally arriving  $M_H$  messages during this slot is given by  $A_H(x_H)$ . On the other hand, that of internally arriving  $M_H$  messages is given by  $B_H(x_H)$ . Each arrival generates an independent and identically distributed busy period for  $M_H$  messages. Thus we get the recurrence relation

$$\Theta_{H,1}(z) = z A_H(\Theta_{H,1}(z)) B_H(\Theta_{H,1}(z)). \tag{3.1}$$

At the beginning of a successive sequence of BH-slots, the number of  $M_H$  messages is distributed as  $(A_H(x_H) - A_H(0))/(1 - A_H(0))$ . Then  $\Theta_{BH}(z)$ , the PGF for the number of successive BH-slots, is given as

$$\Theta_{BH}(z) = \frac{A_H(\Theta_{H,1}(z)) - A_H(0)}{1 - A_H(0)}. \tag{3.2}$$

In GN-cycle, the number of slots in which the server works for  $M_H$  messages is zero when no  $M_H$  message arrives to the system during the first slot, in which an  $M_N$  message is served. This event occurs with probability  $A_H(0)$ . Otherwise, the GN-cycle continues until all  $M_H$  messages in the system are served. In this case the GN-cycle consists of one slot for the service for  $M_N$  message and a some number of slots distributed according to  $\Theta_{BH}(z)$ . From the argument above, the PGF for the number of slots in GN-cycle is given as

$$\text{GN}(z) = \{A_H(0) + (1 - A_H(0))\Theta_{BH}(z)\}z = \{A_H(\Theta_{H,1}(z))\}z. \tag{3.3}$$

### 3.1. Derivation of $\Pi^{FB}(x_H)$

By adopting each departure point of a single cell of  $M_H$  message in FB-period as a Markov point, we have the following equation

$$\Pi^{FB}(x_H) = \Pi_0^{FB} X^{FB}(x_H, 1) \frac{A_H(x_H)B_H(x_H)}{x_H} + [\Pi^{FB}(x_H) - \Pi_0^{FB}] \frac{A_H(x_H)B_H(x_H)}{x_H} \tag{3.4}$$

where  $\Pi_0^{FB}$  is the probability that the number of  $M_H$  messages is zero at a Markov point and  $X^{FB}(x_H, x_N)$  represents the joint PGF for the queue size of  $M_H$  messages and  $M_N$  messages at the beginning of FB-period. Notice that  $A_H(x_H)B_H(x_H)/x_H$  represents the PGF for the number of  $M_H$  messages which arrive and leave the system in a slot. We derive  $X^{FB}(x_H, x_N)$  in the following. Each FB-period follows either C-period or V-period. In case it follows C-period, at least one  $M_H$  message arrives during C-period. We define  $p_{C-FB}$

as the probability that FB-period follows C-period in one Basic-cycle. On the other hand, in case FB-period follows V-period, neither  $M_H$  message nor  $M_N$  message arrives during C-period but at least one  $M_H$  message arrives during V-period. We define  $p_{V-FB}$  as the probability that FB-period follows V-period in one Basic-cycle. From the above observation they are given as

$$\begin{aligned}
 p_{C-FB} &= \{1 - C(A_H(0)A_N(0))\} \frac{1 - A_H(0)}{1 - A_H(0)A_N(0)} \\
 p_{V-FB} &= C(A_H(0)A_N(0)) \frac{1 - V(A_H(0))}{1 - V(A_H(0)A_N(0))}.
 \end{aligned}
 \tag{3.5}$$

The PGF  $X^{FB}(x_H, x_N)$  consists of  $CFB(x_H, x_N)$  if FB-period follows C-period while it consists of  $VFB(x_H, x_N)$  if FB-period follows V-period. Therefore we have

$$X^{FB}(x_H, x_N) = \frac{p_{C-FB}}{p_{C-FB} + p_{V-FB}} CFB(x_H, x_N) + \frac{p_{V-FB}}{p_{C-FB} + p_{V-FB}} VFB(x_H, x_N). \tag{3.6}$$

Solving (3.4) and determining  $\Pi_0^{FB}$  by the normalization condition  $\Pi^{FB}(1) = 1$ , we have

$$\Pi^{FB}(x_H) = \frac{1 - (\alpha_H + \lambda_H)}{\chi_{FB}} \frac{X^{FB}(x_H, 1) - 1}{x_H - A_H(x_H)B_H(x_H)} A_H(x_H)B_H(x_H) \tag{3.7}$$

where  $\chi_{FB}$  is the mean number of  $M_H$  messages at the beginning of FB-period,  $\frac{d}{dx_H} X^{FB}(x_H, 1) |_{x_H=1}$ .

### 3.2. Derivation of $\Pi^{GN}(x_N)$

By taking Markov points at the end of GN-cycles, we have the following equation

$$\begin{aligned}
 \Pi^{GN}(x_N) &= \Pi_0^{GN} X^{GN}(x_N) \frac{GN(A_N(x_N))B_N(x_N)}{x_N} \\
 &\quad + [\Pi^{GN}(x_N) - \Pi_0^{GN}] \frac{GN(A_N(x_N))B_N(x_N)}{x_N},
 \end{aligned}
 \tag{3.8}$$

where  $\Pi_0^{GN}$  is the probability that the number of  $M_N$  messages is zero at a Markov point and  $X^{GN}(x_N)$  represents the PGF for the queue size of  $M_N$  messages at the beginning of the first GN-cycle in one Basic-cycle. Notice that  $GN(A_N(x_N))B_N(x_N)/x_N$  represents the PGF for the number of  $M_N$  messages which arrive and leave the system in one GN-cycle. We derive  $X^{GN}(x_N)$  in the following. From the beginning of Basic-cycle to the beginning of the first GN-cycle in it, the system state changes in four ways.

[Case 1] C-period  $\rightarrow$  GN-cycle

[Case 2] C-period  $\rightarrow$  FB-period  $\rightarrow$  GN-cycle

[Case 3] C-period  $\rightarrow$  V-period  $\rightarrow$  GN-cycle

[Case 4] C-period  $\rightarrow$  V-period  $\rightarrow$  FB-period  $\rightarrow$  GN-cycle

We define  $p_{GN-1}, \dots, p_{GN-4}$  as the probability that the first GN-cycle appears in the corresponding manner described above. We also define  $X_1^{GN}(x_N), \dots, X_4^{GN}(x_N)$  as the PGF for the number of  $M_N$  messages at the beginning of the first GN-cycle which appears in the corresponding manner described above. In Case 1, no  $M_H$  message and at least one  $M_N$  message arrives during the last slot in C-period. In Case 2, at least one  $M_H$  message arrives during the last slot in C-period and at least one  $M_N$  message is in the queue at the end of



FB-period. In Case 3, neither  $M_H$  nor  $M_N$  message arrives during C-period, but at least one  $M_N$  message and no  $M_H$  message arrives during V-period. In Case 4, neither  $M_H$  nor  $M_N$  message arrives during C-period, at least one  $M_H$  message arrives during V-period and at least one  $M_N$  message is in the queue at the end of FB-period. From these observations we have

$$\begin{aligned}
 p_{GN-1} &= \{1 - C(A_H(0)A_N(0))\} \frac{(1 - A_N(0))A_H(0)}{1 - A_H(0)A_N(0)}, \\
 p_{GN-2} &= \{1 - C(A_H(0)A_N(0))\} \frac{1 - A_H(0)}{1 - A_H(0)A_N(0)} \{1 - \text{CFB}(\Theta_{H,1}(A_N(0)), 0)\}, \\
 p_{GN-3} &= C(A_H(0)A_N(0)) \frac{V(A_H(0)) - V(A_H(0)A_N(0))}{1 - V(A_H(0)A_N(0))}, \\
 p_{GN-4} &= C(A_H(0)A_N(0)) \frac{1 - V(A_H(0))}{1 - V(A_H(0)A_N(0))} \{1 - \text{VFB}(\Theta_{H,1}(A_N(0)), 0)\},
 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
 X_1^{GN}(x_N) &= \frac{A_N(x_N) - A_N(0)}{1 - A_N(0)}, \\
 X_2^{GN}(x_N) &= \frac{\text{CFB}(\Theta_{H,1}(A_N(x_N)), x_N) - \text{CFB}(\Theta_{H,1}(A_N(0)), 0)}{1 - \text{CFB}(\Theta_{H,1}(A_N(0)), 0)}, \\
 X_3^{GN}(x_N) &= \frac{V(A_N(x_N)) - V(A_N(0))}{1 - V(A_N(0))}, \\
 X_4^{GN}(x_N) &= \frac{\text{VFB}(\Theta_{H,1}(A_N(x_N)), x_N) - \text{VFB}(\Theta_{H,1}(A_N(0)), 0)}{1 - \text{VFB}(\Theta_{H,1}(A_N(0)), 0)}.
 \end{aligned} \tag{3.10}$$

Finally we have

$$X^{GN}(x_N) = \frac{p_{GN-1}X_1^{GN}(x_N) + p_{GN-2}X_2^{GN}(x_N) + p_{GN-3}X_3^{GN}(x_N) + p_{GN-4}X_4^{GN}(x_N)}{p_{GN-1} + p_{GN-2} + p_{GN-3} + p_{GN-4}}. \tag{3.11}$$

Solving (3.8) and determining  $\Pi_0^{GN}$  by normalization condition  $\Pi^{GN}(1) = 1$ , we have

$$\Pi^{GN}(x_N) = \frac{1 - (\alpha_N + \theta_{GN}\lambda_N)}{\chi_{GN}} \frac{X^{GN}(x_N) - 1}{x_N - \text{GN}(A_N(x_N))B_N(x_N)} \text{GN}(A_N(x_N))B_N(x_N), \tag{3.12}$$

where  $\chi_{GN}$  is the mean number of  $M_N$  messages at the beginning of the first GN-cycle,  $\frac{d}{dx_N} X^{GN}(x_N) |_{x_N=1}$ , and  $\theta_{GN}$  is the mean number of slots in one GN-cycle,  $\frac{d}{dz} \text{GN}(z) |_{z=1}$ .

### 3.3. Derivation of $\Pi^{BH}(x_H)$

By adopting the end of each slot in BH-slot as a Markov point, we have the following equation

$$\begin{aligned}
 \Pi^{BH}(x_H) &= \Pi_0^{BH} \frac{A_H(x_H) - A_H(0)}{1 - A_H(0)} \frac{A_H(x_H)B_H(x_H)}{x_H} \\
 &\quad + [\Pi^{BH}(x_H) - \Pi_0^{BH}] \frac{A_H(x_H)B_H(x_H)}{x_H},
 \end{aligned} \tag{3.13}$$

where  $\Pi_0^{BH}$  is the probability that the number of  $M_H$  messages is zero at a Markov point. Solving (3.13) and determining  $\Pi_0^{BH}$  by normalization condition  $\Pi^{BH}(1) = 1$ , we have

$$\Pi^{BH}(x_H) = \frac{1 - (\alpha_H + \lambda_H)}{\lambda_H} \frac{A_H(x_H) - 1}{x_H - A_H(x_H)B_H(x_H)} A_H(x_H)B_H(x_H). \quad (3.14)$$

### Remark

Although we can derive the PGF for the number of messages at an arbitrary point, we do not present the result due to the space limitation.  $\Pi^{FB}(x_H)$ ,  $\Pi^{GN}(x_N)$  and  $\Pi^{BH}(x_H)$  suffice to derive the waiting time distribution of an arbitrary message.

## 4. Waiting Time Analysis

In this section, we analyze the waiting time of the tagged message. To begin with, we introduce  $M_T$ ,  $\alpha_T$ , GT-cycle and TT-cycle in order to treat the case the tagged message is of  $M_H$  type and the case the tagged message is of  $M_N$  type in a unified way. We call those messages belonging to the same class of the tagged message  $M_T$  messages. That is  $M_T$  messages correspond to  $M_H$  messages if the tagged message is  $M_H$  message and  $M_N$  messages otherwise. Similarly we denote by  $\alpha_T$ , the parameter for retrial probabilities,  $\alpha_H$  or  $\alpha_N$  according to the class of the tagged message. GT-cycle and TT-cycle are also defined in this manner.

The tagged message is called waiting when it is in the system but not being served. We define waiting slots as those slots in which the tagged message is waiting and serving slots as those slots in which it is being served. Sojourn slots consist of waiting slots and serving slots. The waiting time of the tagged message is expressed by the number of waiting slots. We classify the waiting time into two types, initial delay and main delay. Initial delay starts with the arrival of the tagged message and ends at the beginning of TT-cycle. Main delay consists of the slots in TT-cycles aside from serving slots and the slots in the last GT-cycle. In the following, the length of main delay in slots is considered.

### 4.1. Main Delay

First we introduce the following notation in order to make clear the structure of  $n$ -th TT-cycle.

- $a_{n,i}$  : the number of externally arriving  $M_T$  messages during the  $i$ -th GT-cycle in the  $n$ -th TT-cycle
- $b_{n,i}$  : the number of internally arriving  $M_T$  messages (feedbacks) during the  $i$ -th GT-cycle in the  $n$ -th TT-cycle
- $s_{n,i}$  : the number of slots in the  $i$ -th GT-cycle in the  $n$ -th TT-cycle
- $d_n$  : the number of GT-cycles in the  $n$ -th TT-cycle
- $q_{n,i}$  : the number of other  $M_T$  messages present at the beginning of the  $i$ -th GT-cycle in the  $n$ -th TT-cycle
- $w_{n,i}$  : the number of waiting slots at the beginning of the  $i$ -th GT-cycle in the  $n$ -th TT-cycle
- $ql_n$  : the number of other  $M_T$  messages present at the beginning of the  $d_n$ -th GT-cycle in the  $n$ -th TT-cycle
- $wl_n$  : the number of waiting slots at the beginning of the  $d_n$ -th GT-cycle in the  $n$ -th TT-cycle

See Figure 5. We define the probability  $p_{n,i}(k, l)$  as

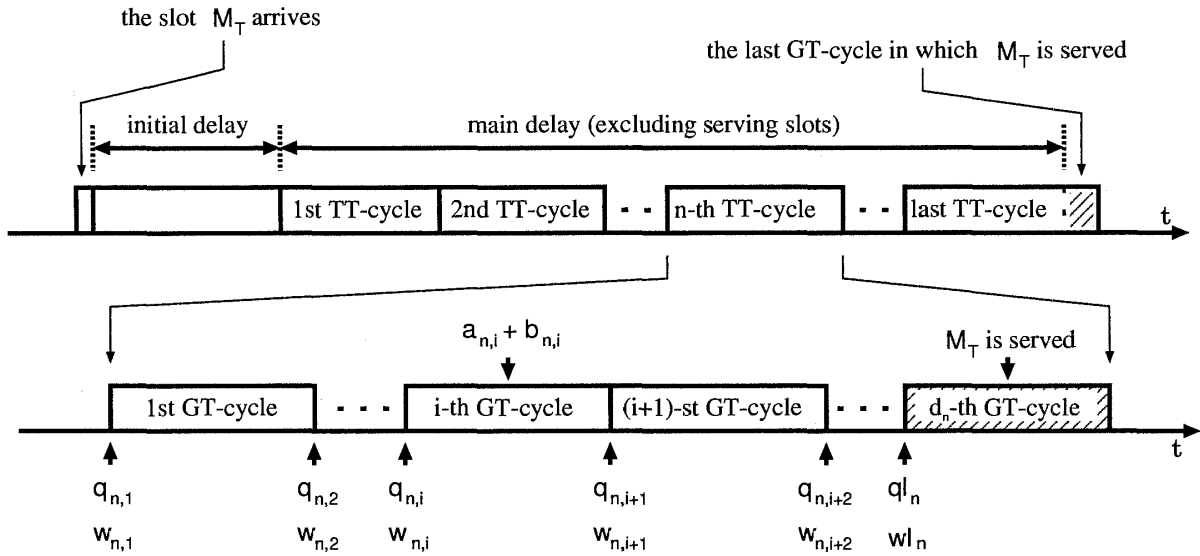


Figure 5: Structure of  $n$ -th TT-cycle

$$p_{n,i}(k, l) \equiv \text{Prob}[d_n \geq i, w_{n,i} = k, q_{n,i} = l].$$

From the total probability result,

$$p_{n,i+1}(k', l') = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \text{Prob}[d_n \geq i + 1, w_{n,i+1} = k', q_{n,i+1} = l' \mid d_n \geq i, w_{n,i} = k, q_{n,i} = l] p_{n,i}(k, l). \quad (4.1)$$

The system remains in the  $n$ -th TT-cycle at least one more GT-cycle if the tagged message is not selected for service at the beginning of the  $i$ -th GT-cycle. This happens with probability  $l/(l+1)$  if  $q_{n,i} = l$ . Under this condition,  $q_{n,i+1} = l'$  if  $a_{n,i} + b_{n,i} = l' - l + 1$ . And  $w_{n,i+1} = k'$  if  $s_{n,i} = k' - k$ . From these observations one can show that

$$\begin{aligned} & \text{Prob}[d_n \geq i + 1, w_{n,i+1} = k', q_{n,i+1} = l' \mid d_n \geq i, w_{n,i} = k, q_{n,i} = l] \\ &= \frac{l}{l+1} \text{Prob}[a_{n,i} + b_{n,i} = l' - l + 1, s_{n,i} = k' - k]. \quad (4.2) \end{aligned}$$

We define  $GT(z)$ ,  $A_T(x_T)$  and  $B_T(x_T)$  as follows

- $GT(z)$  : the PGF for the number of slots in GT-cycle
- $A_T(x_T)$  : the PGF for the number of externally arriving  $M_T$  messages in a slot
- $B_T(x_T)$  : the PGF for the number of internally arriving  $M_T$  messages in one GT-cycle.

After some algebraic manipulation, (4.1) and (4.2) yield

$$H_{n,i+1}(z, x_T) = \frac{GT(zA_T(x_T))}{x_T} B_T(x_T) \left( H_{n,i}(z, x_T) - \frac{1}{x_T} \int_0^{x_T} H_{n,i}(z, y) dy \right), \quad (4.3)$$

where  $H_{n,i}(z, x_T)$  is the joint PGF for the number of waiting slots and other  $M_T$  messages at the beginning of the  $i$ -th GT-cycle in the  $n$ -th T-cycle along with the event that the  $i$ -th GT-cycle exists in the  $n$ -th T-cycle ( $d_n \geq i$ ),

$$H_{n,i}(z, x_T) \equiv \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} p_{n,i}(k, l) z^k x_T^l.$$

The derivation of (4.3) is given in Appendix (A.1). Now, we define the probability  $p'_{n,i}(k, l)$  as

$$p'_{n,i}(k, l) \equiv \text{Prob}[d_n = i, w_{n,i} = k, q_{n,i} = l].$$

The relation between  $p_{n,i}(k, l)$  and  $p'_{n,i}(k, l)$  is

$$p'_n(k, l) = \frac{1}{l+1} p_{n,i}(k, l). \quad (4.4)$$

Next, we define  $I_{n,i}(z, x_T)$ , the joint PGF for the number of waiting slots and other  $M_T$  messages at the beginning of the  $d_n$ -th GT-cycle in the  $n$ -th TT-cycle along with the event the  $i$ -th GT-cycle is the last GT-cycle in the  $n$ -th TT-cycle ( $d_n = i$ ) as

$$I_{n,i}(z, x_T) \equiv \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} p'_{n,i}(k, l) z^k x_T^l.$$

From (4.4), we get

$$I_{n,i}(z, x_T) = \frac{1}{x_T} \int_0^{x_T} H_{n,i}(z, y) dy. \quad (4.5)$$

We define  $I_n(z, x_T)$  as the joint PGF for the number of waiting slots and other  $M_T$  messages at the beginning of the  $d_n$ -th GT-cycle in the  $n$ -th TT-cycle. This is easily given by summing up  $I_{n,i}(z, x_T)$ ,

$$I_n(z, x_T) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \text{Prob}[wl_n = k, ql_n = l] z^k x_T^l = \sum_{i=0}^{\infty} I_{n,i}(z, x_T). \quad (4.6)$$

From (4.3) and (4.5), one can derive

$$I_n(z, x_T) + (x_T - \text{GT}(zA_T(x_T))B_T(x_T)) \frac{\partial}{\partial x_T} I_n(z, x_T) = H_{n,1}(z, x_T). \quad (4.7)$$

The derivation of (4.7) is described in Appendix (A.2).

On the condition that the tagged message requires more than  $n$  slots of service, no message leaves the system at the end of the  $(n-1)$ -st TT-cycle. Therefore  $H_{n,1}(z, x_T)$  is given by

$$H_{n,1}(z, x_T) = \begin{cases} \frac{\text{GT}(zA_T(x_T))}{z} I_{n-1}(z, x_T) & (n \geq 2) \\ U(z, x_T) & (n = 1), \end{cases} \quad (4.8)$$

where  $U(z, x_T)$  is the joint PGF for the number of waiting slots and other  $M_T$  messages at the beginning of the first TT-cycle. We define  $I(z, x_T)$  as the joint PGF for the number of waiting slots and other  $M_T$  messages at the beginning of GT-cycle in which the tagged message is served. Since the number of packets in  $M_T$  message is geometrically distributed with parameter  $\alpha_T$ ,  $I(z, x_T)$  is given as,

$$I(z, x_T) = \sum_{n=1}^{\infty} (1 - \alpha_T) \alpha_T^{n-1} I_n(z, x_T).$$

From (4.7) and (4.8), we have the differential equation for  $I(z, x_T)$ ,

$$\left\{ 1 - \alpha_T \frac{\text{GT}(zA_T(x_T))}{z} \right\} I(z, x_T) + (x_T - \text{GT}(zA_T(x_T))B_T(x_T)) \frac{\partial}{\partial x_T} I(z, x_T) = (1 - \alpha_T)U(z, x_T). \tag{4.9}$$

The derivation of (4.9) is explained in more detail in Appendix (A.3).

### 4.2. Initial Delay

In this section we consider events in initial delay slots for each type of messages classified according to the system state the tagged message observes upon arrival, and obtain the appropriate form of  $U(z, x_T)$  in (4.9) for them.

#### 4.2.1. Initial Delay of the Tagged High-Priority Message

By conditioning the system state which the tagged  $M_H$  message observes upon arrival, we consider the waiting time of  $M_H$  message in five different cases. We denote  $M_H$  messages which arrive to the system in C-period as  $M_H^C$ , V-period as  $M_H^V$ , FB-period as  $M_H^{FB}$ , BN-slots as  $M_H^{BN}$ , BH-slots as  $M_H^{BH}$ . We define  $U_{C,H}(z, x_H)$  as the joint PGF for the number of waiting slots of the tagged  $M_H^C$  message and the number of other  $M_H$  messages at the beginning of the first TH-cycle. Since the first TH-cycle for the tagged  $M_H^C$  message begins immediately after its arrival,  $U_{C,H}(z, x_H)$  does not contain variable  $z$ . However we adopt the expression  $U_{C,H}(z, x_H)$  to unify the notation.  $U_{V,H}(z, x)$ ,  $U_{FB,H}(z, x)$ ,  $U_{BN,H}(z, x)$ , and  $U_{BH,H}(z, x)$ . are defined similarly. Notice that  $\text{GT}(z) = z$  for any  $M_H$  message. First we define  $L_H(x_H)$  as the PGF for the number of other  $M_H$  messages which arrive to the system during the same slot as the tagged  $M_H$  message. It is given as

$$L_H(x_H) = \frac{\frac{d}{dx_H} A_H(x_H)}{\lambda_H}. \tag{4.10}$$

The derivation of (4.10) is shown in Appendix (A.4). The first TH-cycle for the tagged  $M_H^C$  message begins at the next slot after its arrival and other  $M_H$  messages in the queue at that time correspond to those arriving to the system in the same slot as the tagged  $M_H^C$  message. Therefore we have

$$U_{C,H}(z, x_H) = L_H(x_H). \tag{4.11}$$

The tagged  $M_H^V$  message has to wait until the vacation ends. To begin with, we define  $Z_-$  and  $Z_+$  as the number of slots in a vacation already elapsed and remaining at the time the tagged message arrives. (The slot in which the tagged message arrives belongs to neither of them.) Let  $V_{\pm}(z_-, z_+)$  be the PGF for  $Z_-$  and  $Z_+$ , and we can derive it as

$$V_{\pm}(z_-, z_+) = \frac{1}{\frac{d}{dz} V(z) |_{z=1}} \frac{V(z_+) - V(z_-)}{z_+ - z_-}. \tag{4.12}$$

The derivation of (4.12) is explained in Appendix (A.5). Substituting  $A_H(x_H)$  for  $z_-$  and  $zA_H(x_H)$  for  $z_+$  yields the joint PGF for the number of other  $M_H$  messages which arrive to the system during the vacation excluding the slot in which the tagged  $M_H^V$  message arrives and the number of waiting slots at the end of the vacation. Taking into account the messages arriving in the same slot as the tagged message, we have

$$U_{V,H}(z, x_H) = \frac{1}{\frac{d}{dz} V(z) |_{z=1}} \frac{V(A_H(x_H)) - V(zA_H(x_H))}{(A_H(x_H) - zA_H(x_H))} L_H(x_H). \tag{4.13}$$

For the tagged  $M_H^{FB}$  message, the number of waiting slots at the beginning of the first TH-cycle is zero. To find the number of other messages at the beginning of the next slot after arrival, we define  $\Pi^{FB-}(x_H)$  as the PGF for the number of  $M_H$  messages at the beginning of FB-slots aside from the  $M_H$  message which is going to be served.

The  $M_H$  messages which are in the system at the end of FB-slot consist of those having been in the system at the beginning of the FB-slot aside from the  $M_H$  message which is going to be served, those externally arriving to the system in the slot and those internally arriving. The number of them is distributed as  $\Pi^{FB-}(x_H)$ ,  $A_H(x_H)$  and  $B_H(x_H)$ . Thus we have

$$\Pi^{FB-}(x_H)A_H(x_H)B_H(x_H) = \Pi^{FB}(x_H). \quad (4.14)$$

From which we have  $U_{FB,H}(z, x_H)$  as

$$U_{FB,H}(z, x_H) = \Pi^{FB-}(x_H)L_H(x_H)B_H(x_H) = \frac{\Pi^{FB}(x_H)}{A_H(x_H)}L_H(x_H). \quad (4.15)$$

**Note:**  $\Pi^{FB}(x_H)$  is given by (3.7) in Section 3.

For  $M_H^{BN}$  and  $M_H^{BH}$ , we can make an argument similar to those for  $M_H^C$  and  $M_H^{FB}$  respectively. Then we get

$$U_{BN,H}(z, x_H) = L_H(x_H) \quad (4.16)$$

and

$$U_{BH,H}(z, x_H) = \frac{\Pi^{BH}(x_H)}{A_H(x_H)}L_H(x_H). \quad (4.17)$$

**Note:**  $\Pi^{BH}(x_H)$  is given by (3.14) in Section 3.

#### 4.2.2. Initial Delay of the Tagged Normal-Priority Message

By conditioning on the system state that an arriving  $M_N$  message observes, we consider its initial delay in four cases. When  $M_N$  message arrives to the system during C-period, it is classified as  $M_N^C$  message. We denote  $M_N$  messages which arrive to the system during V-period as  $M_N^V$ , FB-period as  $M_N^{FB}$ , GN-cycles as  $M_N^{GN}$ .

In the following we derive the joint PGF for the number of waiting slots of the tagged  $M_N^C$  ( $M_N^V$ ,  $M_N^{FB}$ ,  $M_N^{GN}$ ) message and the number of other  $M_N$  messages at the beginning of the first TN-cycle,  $U_{C,N}(z, x_N)$  ( $U_{V,N}(z, x_N)$ ,  $U_{FB,N}(z, x_N)$ ,  $U_{GN,N}(z, x_N)$ ). For the tagged  $M_N^C$  message, the first TN-cycle begins after FB-period initiated by  $M_H$  messages which arrive in the same slot as the tagged  $M_N^C$  message ends. The number of such  $M_H$  messages is distributed as  $A_H(x_H)$ . Therefore the number of waiting slots and the number of  $M_N$  messages which arrive during the FB-period is distributed as  $A_H(\Theta_1(zA_H(x_H)))$  at the beginning of the first TN-cycle. We define  $L_N(x_N)$  as the PGF for the number of messages which arrive to the system in the same slot as the tagged  $M_N$  message.  $L_N(x_N)$  is derived by the similar argument to (4.10) and it is given as

$$L_N(x_N) = \frac{\frac{d}{dx_N}A_N(x_N)}{\lambda_N}. \quad (4.18)$$

Using (4.18), we have

$$U_{C,N}(z, x_N) = A_H(\Theta_1(zA_N(x_N)))L_N(x_N). \quad (4.19)$$

To find  $U_{V,N}(z, x_N)$ , let us introduce the joint PGF  $VE(z, x_H, x_N)$  for the number of waiting slots and the queue size of each class at the end of a vacation. By a similar argument to (4.13), we have

$$VE(z, x_H, x_N) = \frac{1}{\frac{d}{dz}V(z)|_{z=1}} \frac{V(A_H(x_H)A_N(x_N)) - V(zA_H(x_H)A_N(x_N))}{A_H(x_H)A_N(x_N) - zA_H(x_H)A_N(x_N)} A_H(x_H)L_N(x_N). \quad (4.20)$$

The number of waiting slots is increased by one and the number of  $M_N$  messages is increased by the number distributed as  $A_N(x_N)$  in every FB-slot. Thus we substitute  $\Theta_1(zA_N(x_N))$  for  $x_H$  in  $VE(z, x_H, x_N)$  and get

$$U_{V,N}(z, x_N) = VE(z, \Theta_{H,1}(zA_N(x_N)), x_N). \quad (4.21)$$

Next we consider the tagged  $M_N^{FB}$  message. In this case, the numbers of  $M_H$  messages and  $M_N$  messages are distributed as  $X^{FB}(x_H, x_N)$  at the beginning of FB-period. The PGF for the number of waiting slots and other  $M_N$  messages at the end of the FB-period, which corresponds to  $U_{FB,N}(z, x_N)$ , is given by

$$U_{FB,N}(z, x_N) = \frac{X^{FB}(\Theta_{H,1}(A_N(x_N)), x_N) - X^{FB}(\Theta_{H,1}(zA_N(x_N)), x_N)}{\theta_{H,1}\chi_{FB}(1-z)} \frac{L_N(x_N)}{A_N(x_N)}. \quad (4.22)$$

where  $\theta_{H,1}$  is the mean number of slots the server spends before it serves all  $M_H$  messages in the queue on the condition that one  $M_H$  message exists in the system at the beginning of a slot,  $\frac{d}{dz}\Theta_{H,1}(z)|_{z=1}$ . The derivation of (4.22) is described in detail in Appendix (A.6). For the tagged  $M_N^{GN}$  message, we can make a similar argument to (4.13) and (4.15), and have

$$U_{GN,N}(z, x_N) = \frac{\Pi_{GN}(x_N)}{\theta_{GN}\text{GN}(A_N(x))} \frac{\text{GN}(A_N(x_N)) - \text{GN}(zA_N(x_N))}{A_N(x_N) - zA_N(x_N)} L_N(x_N). \quad (4.23)$$

**Note:**  $\Pi^{GN}(x_N)$  is given by (3.12) in Section 3.

### 4.3. Unconditional Waiting Time

In the following subsections, we derive the unconditional  $U(z, x)$  for  $M_H$  messages and  $M_N$  messages. To do so, we first obtain the probabilities that the system is in each period and cycle introduced in Section 2.1, at a random point in time. Since we assume the number of  $M_H$  and  $M_N$  messages which arrive to the system in a slot is distributed identically and independently from slot to slot, we can derive the unconditional waiting time from the above probabilities.

#### 4.3.1. Mean Number of Slots in Each Period and Cycle

We define  $C_L(z)$  as the number of slots in C-period. C-period ends in two ways,

[Case 1] Upon messages arrivals(including the case Close-time ends at the same time)

[Case 2] Close-time ends

In Case 1, the number of slots in C-period is  $k$  when the following three events happens. There is no message arrives in the 1-st,  $\dots$ ,  $(k-1)$ -st slots in C-period,  $M_H$  messages or  $M_N$  messages arrive in the  $k$ -th slot and close-time does not ends by  $k$ -th slots in C-period.

On the other hand in Case 2 the number of slots in C-period is  $k$  when there is no message arrives in 1-st,  $\dots$ ,  $k$ -th slots and close-time ends  $k$ -th slots. From above observations, we have

$$\begin{aligned} & \text{Prob}[\text{the number of slots in C-period is } k] \\ &= (A_H(0)A_N(0))^{k-1}(1 - A_H(0)A_N(0)) \left(1 - \sum_{l=0}^{k-1} c_l\right) + (A_H(0)A_N(0))^k c_k. \end{aligned} \quad (4.24)$$

where  $c_l$  is the probability that the number of slots in close-time is  $l$ ,  $\frac{1}{l!}(\frac{d}{dz})^l C(z) |_{z=0}$ . From (4.24), we have

$$C_L(z) = \frac{(1 - A_H(0)A_N(0))z}{1 - zA_H(0)A_N(0)}(1 - C(zA_H(0)A_N(0))) + C(zA_H(0)A_N(0)). \quad (4.25)$$

The derivation of (4.25) is in Appendix(A.7). The mean number of slots in C-period,  $N_C$ , is given as

$$N_C = \frac{d}{dz} C_L(z) |_{z=1} = \frac{1 - C(A_H(0)A_N(0))}{1 - A_H(0)A_N(0)}.$$

We define  $V_L(z)$  as the number of slots in V-period. This is given as

$$V_L(z) = \frac{V(z) - V(zA_H(0)A_N(0))}{1 - V(zA_H(0)A_N(0))}. \quad (4.26)$$

The derivation of (4.26) is in Appendix(A.8). The mean number of slots in V-period,  $N_V$ , is given as

$$N_V = \frac{d}{dz} V_L(z) |_{z=1} = \frac{V_m}{1 - V(A_H(0)A_N(0))}.$$

where  $V_m$  is the mean number of slots in a vacation,  $\frac{d}{dz} V(z) |_{z=1}$ . The mean number of slots in FB-period,  $N_{FB}$ , is given as

$$N_{FB} = \frac{d}{dz} X^{FB}(\Theta_{H,1}(z), 1) |_{z=1}.$$

The mean number of slots in GN-cycles in one Basic-cycle,  $N_{GN}$ , is given as

$$N_{GN} = \frac{d}{dz} X^{GN}(\Theta_{N,1}(z)) |_{z=1}$$

where  $\Theta_{N,1}(z)$  is the PGF for the number of slots the server spends before it serves all  $M_H$  and  $M_N$  messages in the queue on the condition that no  $M_H$  message and one  $M_N$  message exists in the system at the beginning of a slot. This is given by the similar argument to  $\Theta_{H,1}(z)$ ,

$$\Theta_{N,1}(z) = \text{GN}(zA_N(\Theta_{N,1}(z)))B_N(\Theta_{N,1}(z)).$$

The mean number of BN-slots in one Basic-cycle,  $N_{BN}$ , corresponds to the number of  $M_N$  messages served in one Basic-cycle. This is given by  $N_{GN}/\theta_{GN}$ . Since GN-cycles consist of BH-slots and BN-slots, the mean number of BN-slots in one Basic-cycle  $N_{BH}$  is given by  $N_{GN} - N_{BN}$ .

We next consider  $N_{BC}$ , the mean number of slots in one Basic-cycle. From the beginning of Basic-cycle to the beginning of the next Basic-cycle, the system state changes in six ways.



- [Case 1] C-period  $\rightarrow$  GN-cycle  $\rightarrow$  end  
 [Case 2] C-period  $\rightarrow$  FB-period  $\rightarrow$  GN-cycle  $\rightarrow$  end  
 [Case 3] C-period  $\rightarrow$  V-period  $\rightarrow$  GN-cycle  $\rightarrow$  end  
 [Case 4] C-period  $\rightarrow$  V-period  $\rightarrow$  FB-period  $\rightarrow$  GN-cycle  $\rightarrow$  end  
 [Case 5] C-period  $\rightarrow$  FB-period  $\rightarrow$  end  
 [Case 6] C-period  $\rightarrow$  V-period  $\rightarrow$  FB-period  $\rightarrow$  end

We define  $p_{BC-1}, \dots, p_{BC-6}$  as the probability that the system state changes in the corresponding manner described above. The probabilities  $p_{BC-1}, \dots, p_{BC-4}$  consist with  $p_{GN-1}, \dots, p_{GN-4}$  defined in Section 3, and  $p_{BC-5}$  and  $p_{BC-6}$  are given as

$$\begin{aligned} p_{BC-5} &= \{1 - C(A_H(0)A_N(0))\} \frac{1 - A_H(0)}{1 - A_H(0)A_N(0)} \text{CFB}(\Theta_{H,1}(A_N(0)), 0) \\ p_{BC-6} &= C(A_H(0)A_N(0)) \frac{1 - V(A_H(0))}{1 - V(A_H(0)A_N(0))} \text{VFB}(\Theta_{H,1}(A_N(0)), 0). \end{aligned} \quad (4.27)$$

Using these results, we have

$$\begin{aligned} N_{BC} &= N_C + \text{Prob}[V\text{-period exists in Basic-cycle}]N_V \\ &+ \text{Prob}[FB\text{-period exists in Basic-cycle}]N_{FB} + \text{Prob}[GN\text{-cycles exist in Basic-cycle}]N_{GN} \\ &= N_C + (p_{BC-3} + p_{BC-4} + p_{BC-6})N_V + (p_{BC-2} + p_{BC-4} + p_{BC-5} + p_{BC-6})N_{FB} \\ &\quad + (p_{BC-1} + p_{BC-2} + p_{BC-3} + p_{BC-4})N_{GN}. \end{aligned} \quad (4.28)$$

#### 4.3.2. System State at Random Points in Time

Since those epochs when the system becomes empty, the beginnings of Basic-cycle, are regenerative points, we can obtain the probabilities that the system is in each period and cycle at a random point in time, Prob[C-period], Prob[V-period], Prob[FB-period], Prob[GN-cycles], Prob[BN-slots] and Prob[BH-slots] as (sec. 6.4 in Heyman and Sobel [4])

$$\begin{aligned} \text{Prob}[C\text{-period}] &= \frac{N_C}{N_{BC}} \\ \text{Prob}[V\text{-period}] &= \frac{N_V}{N_{BC}}(p_{BC-3} + p_{BC-4} + p_{BC-6}) \\ \text{Prob}[FB\text{-period}] &= \frac{N_{FB}}{N_{BC}}(p_{BC-2} + p_{BC-4} + p_{BC-5} + p_{BC-6}) \\ \text{Prob}[GN\text{-cycles}] &= \frac{N_{GN}}{N_{BC}}(p_{BC-1} + p_{BC-2} + p_{BC-3} + p_{BC-4}) \\ \text{Prob}[BN\text{-slots}] &= \frac{N_{BN}}{N_{BC}}(p_{BC-1} + p_{BC-2} + p_{BC-3} + p_{BC-4}) \\ \text{Prob}[BH\text{-slots}] &= \frac{N_{BH}}{N_{BC}}(p_{BC-1} + p_{BC-2} + p_{BC-3} + p_{BC-4}). \end{aligned} \quad (4.29)$$

#### 4.3.3. Differential Equation for Unconditional Waiting Time Analysis

We can derive the unconditional  $U(z, x)$  for  $M_H$  messages,  $U_H(z, x_H)$ , as

$$\begin{aligned} U_H(z, x_H) &= \text{Prob}[C\text{-period}]U_{C,H}(z, x_H) + \text{Prob}[V\text{-period}]U_{V,H}(z, x_H) \\ &\quad + \text{Prob}[FB\text{-period}]U_{FB,H}(z, x_H) + \text{Prob}[BN\text{-slots}]U_{BN,H}(z, x_H) \\ &\quad + \text{Prob}[BH\text{-slots}]U_{BH,H}(z, x_H) \end{aligned} \quad (4.30)$$

and the unconditional  $U(z, x)$  for  $M_N$  messages,  $U_N(z, x_H)$ , as

$$U_N(z, x_N) = \text{Prob}[\text{C-period}]U_{C,N}(z, x_N) + \text{Prob}[\text{V-period}]U_{V,N}(z, x_N) \\ + \text{Prob}[\text{FB-period}]U_{FB,N}(z, x_N) + \text{Prob}[\text{GN-cycles}]U_{GN,N}(z, x_N). \quad (4.31)$$

From (4.30), (4.31) and (4.9), we have the differential equation for the PGF for the number of waiting slots and the number of other messages at the beginning of the last GH- or GN-cycle,  $I_H(z, x_H)$  and  $I_N(z, x_N)$ , as

$$\left\{ 1 - \alpha_H \frac{\text{GH}(zA_H(x_H))}{z} \right\} I_H(z, x_H) + (x_H - \text{GH}(zA_H(x_H)))B_H(x_H) \frac{\partial}{\partial x_H} I_H(z, x_H) = \\ (1 - \alpha_H)U_H(z, x_H). \quad (4.32)$$

and

$$\left\{ 1 - \alpha_N \frac{\text{GN}(zA_N(x_N))}{z} \right\} I_N(z, x_N) + (x_N - \text{GN}(zA_N(x_N)))B_N(x_N) \frac{\partial}{\partial x_N} I_N(z, x_N) = \\ (1 - \alpha_N)U_N(z, x_N). \quad (4.33)$$

#### 4.4. Mean Waiting Time

In this section, we derive the mean waiting time for  $M_H$  messages. Differentiating (4.32) by  $x_H$  and substituting  $z = x_H = 1$  yield

$$\frac{\partial}{\partial x_H} I_H(z, x_H) \Big|_{z=x_H=1} = \frac{(1 - \alpha_H) \frac{\partial}{\partial x_H} U(z, x_H) \Big|_{z=x_H=1} + \alpha_H \lambda_H}{2 - \lambda_H - 2\alpha_H}. \quad (4.34)$$

Differentiating (4.32) by  $z$  and substituting  $z = x_H = 1$  yield

$$\frac{\partial}{\partial z} I_H(z, x_H) \Big|_{z=x_H=1} = \frac{\partial}{\partial z} U(z, x_H) \Big|_{z=x_H=1} + \frac{\frac{\partial}{\partial x_H} I_H(z, x_H) \Big|_{z=x_H=1}}{1 - \alpha_H}. \quad (4.35)$$

Since we have  $U(z, x_H)$  explicitly, we can obtain the mean waiting time for  $M_H$  messages,  $\frac{\partial}{\partial z} I_H(z, x_H) \Big|_{z=x_H=1}$ , from (4.34) and (4.35). Higher moments of the waiting time are obtained in a similar way.

### 5. Numerical Results

In this section, we present some numerical results using commercially available mathematical software *Maple V R4*.

Under the following scenario, we have plotted the mean waiting times for  $M_H$  messages in Figure 6 and those for  $M_N$  messages in Figure 7 as a function of the length of close-time  $C$  for different lengths of vacation  $V$ .

- $A_H(x_H) = 0.9 + 0.02x_H + 0.02x_H^2 + 0.02x_H^3 + 0.02x_H^4 + 0.02x_H^5$
- $A_N(x_N) = 0.9 + 0.02x_N + 0.02x_N^2 + 0.02x_N^3 + 0.02x_N^4 + 0.02x_N^5$
- $\alpha_H = \alpha_N = 0.25$ .

In Figure 6, we observe that the mean waiting times for  $M_H$  messages get shorter as  $C$  increases in general but that it behaves in a different way for  $V = 1$ . This comes from the fact that our system is reduced to a priority system with neither close-time nor vacations under  $V = 1$  and the length of close-time in no way affects the behavior of the system. The similar argument holds in Figure 7 except that the mean waiting times for  $M_N$  messages are much higher than those for  $M_H$  messages.

In Figure 8, we have plotted the mean waiting times for  $M_H$  messages as a function of  $C$  under the following scenario.

- $A_N(x_N) = 0.9 + 0.02x_N + 0.02x_N^2 + 0.02x_N^3 + 0.02x_N^4 + 0.02x_N^5$
- $\alpha_H = \alpha_N = 0.25$
- $V = 10$
- $A_H(x_H) = \begin{cases} 0.9 + 0.02x_H + 0.02x_H^2 + 0.02x_H^3 + 0.02x_H^4 + 0.02x_H^5 & \text{Case 1} \\ 0.95 + 0.01x_H + 0.01x_H^2 + 0.01x_H^3 + 0.01x_H^4 + 0.01x_H^5 & \text{Case 2} \\ 0.99 + 0.002x_H + 0.002x_H^2 + 0.002x_H^3 + 0.002x_H^4 + 0.002x_H^5 & \text{Case 3.} \end{cases}$

We observe that the mean waiting times for  $M_H$  messages get shorter as  $C$  increases in general, and those for different  $A_H(x_H)$  are plotted in reverse order at  $C = 1$  and  $C = 1000$ . This comes from the fact in the following. When  $C$  is very large, the occurrence of vacations is rare. The mean waiting times get longer as the number of arriving  $M_H$  messages becomes larger. On the other hand, when  $C$  is not very large, the appearance of vacations is less rare. As the probability that no  $M_H$  message arrives in a slot becomes larger, the probability that close-time expires increases. This leads to frequent occurrence of vacation periods and increase in the mean waiting time. The similar phenomenon is also reported in [9].

In Figure 9, we have plotted the mean waiting times for  $M_H$  messages as a function of  $C$  under the following scenario.

- $A_H(x_H) = 0.9 + 0.02x_H + 0.02x_H^2 + 0.02x_H^3 + 0.02x_H^4 + 0.02x_H^5$
- $\alpha_H = \alpha_N = 0.25$
- $V = 10$
- $A_N(x_N) = \begin{cases} 0.99 + 0.002x_N + 0.002x_N^2 + 0.002x_N^3 + 0.002x_N^4 + 0.002x_N^5 & \text{Case 1} \\ 0.95 + 0.01x_N + 0.01x_N^2 + 0.01x_N^3 + 0.01x_N^4 + 0.01x_N^5 & \text{Case 2} \\ 0.9 + 0.02x_N + 0.02x_N^2 + 0.02x_N^3 + 0.02x_N^4 + 0.02x_N^5 & \text{Case 3.} \end{cases}$

Performance measures of high-priority-messages are not affected by the behavior of the normal-priority-messages in preemptive service discipline generally. In our system, however, close-time tends to be suspended by arrivals of  $M_N$  messages and vacation period less frequently occurs. Therefore, as the arrival rate of  $M_N$  messages becomes higher the mean waiting time for  $M_H$  messages becomes lower.

### Remark

For a special case  $C = 0$ , our numerical results are found to be coincident with that under FCFS in [11].

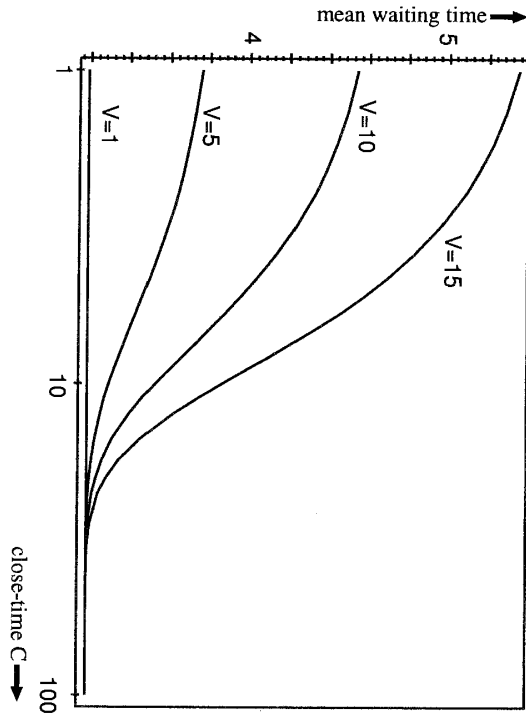


Figure 6  
Mean waiting times for  $M_H$  messages with different value for  $V$

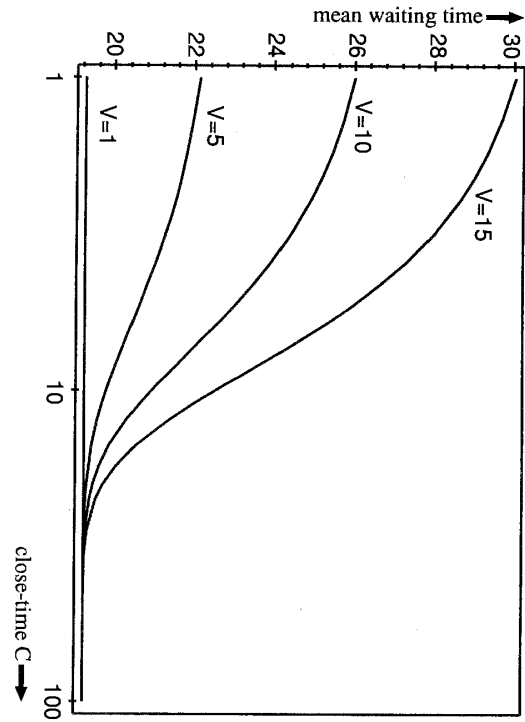


Figure 7  
Mean waiting times for  $M_N$  messages with different value for  $V$

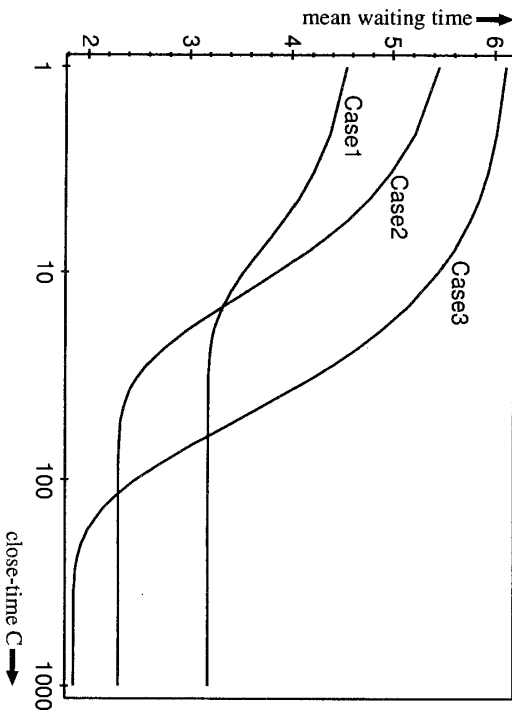


Figure 8  
Mean waiting times for  $M_H$  messages with different  $A_H(x_H)$

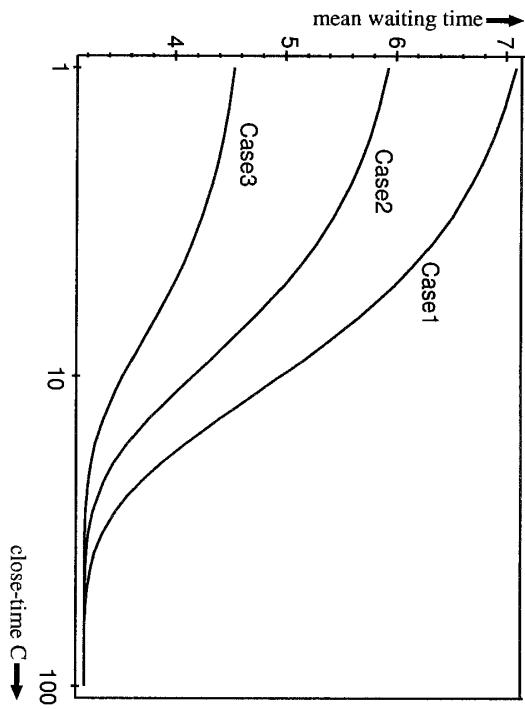


Figure 9  
Mean waiting times for  $M_H$  messages with different  $A_N(x_N)$

## References

- [1] H. Bruneel and B.G. Kim: *Discrete-time Models for Communication Systems Including ATM* (Kluwer Academic Publishers, Boston, 1993).
- [2] T.M. Chen and S.S. Liu: *ATM Switching Systems* (Artech House Publishers, Boston, 1995).
- [3] B.T. Doshi: Queueing systems with vacations - A survey. *Queueing Systems*, 1 (1986) 29-66.
- [4] D.P. Heyman and M.J. Sobel: *Stochastic Models in Operations Research, volume 1: Stochastic Processes and Operating Characteristics* (McGraw-Hill Publishing Company, New York, 1982).
- [5] O. Kella and U. Yechiali: Priorities in M/G/1 queue with server vacations. *Naval Research Logistics*, 35 (1988) 23-34.
- [6] K. Laevens and H. Bruneel: Delay analysis for ATM queue with random order of service. *Electronics Letters*, 31 (1995) 346-347.
- [7] K. Laevens and H. Bruneel: Discrete-time queueing models with feedback for input-buffered ATM switches. *Performance Evaluation*, 27 & 28 (1996) 71-87.
- [8] K. Laevens and H. Bruneel: Waiting times in ATM multiplexers with random order of service. working paper.
- [9] Y. Sakai, Y. Takahashi, Y. Takahashi and T. Hasegawa: A composite queue with vacation/set-up/close-down times for IP over ATM networks. *Journal of the Operations Research Society of Japan*, 41 (1998) 68-80.
- [10] H. Takagi: *Queueing Analysis, volume 1: Vacation and Priority Systems* (Elsevier Science Publishers B.V., Amsterdam, 1991).
- [11] H. Takagi: *Queueing Analysis, volume 3: Discrete-time Systems* (Elsevier Science Publishers B.V., Amsterdam, 1993).
- [12] A.S. Tanenbaum: *Computer Networks Third Edition* (Prentice Hall Inc., New Jersey, 1996).

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## A. Appendix

### A.1. Derivation of equation (4.3)

We begin with equation

$$p_{n,i+1}(k', l') = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{l}{l+1} p_{n,i}(k, l) \text{Prob}[s_i = k' - k, a_i + b_i = l' - l + 1]. \quad (\text{A.1})$$

By summing up (A.1), we have

$$\begin{aligned} & \sum_{k'=0}^{\infty} \sum_{l'=0}^{\infty} p_{n,i+1}(k', l') z^{k'} x_T^{l'} \\ &= \frac{1}{x_T} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{l}{l+1} p_{n,i}(k, l) z^k x_T^l \\ & \quad \times \sum_{k'-k=0}^{\infty} \sum_{l'-l+1=0}^{\infty} \text{Prob}[s_i = k' - k, a_i + b_i = l' - l + 1] z^{k'-k} x_T^{l'-l+1}. \end{aligned} \quad (\text{A.2})$$

The l.h.s of (A.2) equals  $H_{n,i+1}(z, x_T)$  and the r.h.s of (A.2) is considered in the following.

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{l}{l+1} p_{n,i}(k, l) z^k x_T^l \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(1 - \frac{1}{l+1}\right) p_{n,i}(k, l) z^k x_T^l = H_{n,i}(z, x_T) - \frac{1}{x_T} \int_0^{x_T} H_{n,i}(z, y) dy. \end{aligned} \quad (\text{A.3})$$

On the condition  $s_i = k' - k$ , the PGF for the number of  $M_T$  messages which arrive in the  $i$ -th GT-cycle is given by  $(A_T(x_T))^{k'-k} B_T(x_T)$ . Therefore

$$\sum_{k'-k=0}^{\infty} \sum_{l'-l+1=0}^{\infty} \text{Prob}[s_i = k' - k, a_i + b_i = l' - l + 1] z^{k'-k} x_T^{l'-l+1} = \text{GT}(z A_T(x_T)) B_T(x_T). \quad (\text{A.4})$$

From (A.3) and (A.4), we have

$$H_{n,i+1}(z, x_T) = \frac{\text{GT}(z A_T(x_T)) B_T(x_T)}{x_T} \left( H_{n,i}(z, x_T) - \frac{1}{x_T} \int_0^{x_T} H_{n,i}(z, y) dy \right).$$

### A.2. Derivation of equation (4.7)

From equation(4.5) one can derive

$$\frac{\partial}{\partial x_T} I_{n,i}(z, x_T) = \frac{1}{x_T} \left( H_{n,i}(z, x) - \frac{1}{x_T} \int_0^{x_T} H_{n,i}(z, y) dy \right), \quad (\text{A.5})$$

and

$$H_{n,i}(z, x_T) = \frac{\partial}{\partial x_T} (x_T I_{n,i}(z, x_T)) = I_{n,i}(z, x_T) + x_T \frac{\partial}{\partial x_T} I_{n,i}(z, x_T). \quad (\text{A.6})$$

Substituting (A.5) and (A.6) into (4.3) yields

$$I_{n,i+1}(z, x_T) + x_T \frac{\partial}{\partial x_T} I_{n,i+1}(z, x_T) = \text{GT}(z A_T(x_T)) B_T(x_T) \frac{\partial}{\partial x_T} I_{n,i}(z, x_T). \quad (\text{A.7})$$

Summing up (A.7) for  $i \geq 1$  and

$$I_{n,1}(z, x_T) + x_T \frac{\partial}{\partial x_T} I_{n,1}(z, x_T) = H_{n,1}(z, x_T),$$

yields

$$I_n(z, x_T) + \text{GT}(x_T - (zA(x_T))B(x_T)) \frac{\partial}{\partial x_T} I_n(z, x_T) = H_{n,1}(z, x_T).$$

### A.3. Derivation of equation (4.9)

From (4.7) and (4.8), we have the following set of equations.

$$\begin{aligned} I_1(z, x_T) + (x_T - \text{GT}(zA(x_T))B(x_T)) \frac{\partial}{\partial x_T} I_1(z, x_T) &= U(z, x_T) \\ I_2(z, x_T) + (x_T - \text{GT}(zA(x_T))B(x_T)) \frac{\partial}{\partial x_T} I_2(z, x_T) &= \frac{\text{GT}(zA(x_T))}{z} I_1(z, x_T) \\ I_3(z, x_T) + (x_T - \text{GT}(zA(x_T))B(x_T)) \frac{\partial}{\partial x_T} I_3(z, x_T) &= \frac{\text{GT}(zA(x_T))}{z} I_2(z, x_T) \\ &\vdots \end{aligned}$$

We multiply  $\alpha_T^{i-1}(1-\alpha_T)$  on the both sides of the  $i$ -th equation and sum up all the equations. Then we have

$$\begin{aligned} I(z, x_T) + (x_T - \text{GT}(zA(x_T))B(x_T)) \frac{\partial}{\partial x_T} I(z, x_T) \\ = \alpha_T \frac{\text{GT}(zA(x_T))}{z} I(z, x_T) + (1 - \alpha_T) U(z, x_T) \end{aligned}$$

Then (4.9) follows.

### A.4. Derivation of equation (4.10)

We denote by  $L$  the number of other  $M_H$  messages which arrive to the system during the same slot as the tagged  $M_H$  message and by  $a_H$  the number of new arrivals of  $M_H$  messages in a slot. From the idea of number-biased sampling, we have

$$\text{Prob}[L = l] \propto (l + 1) \text{Prob}[a_H = l + 1].$$

Therefore

$$\text{Prob}[L = l] = \frac{(l + 1) \text{Prob}[a_H = l + 1]}{\sum_{k=0}^{\infty} ((k + 1) \text{Prob}[a_H = k + 1])} = \frac{(l + 1) \text{Prob}[a_H = l + 1]}{\frac{d}{dx_H} A_H(x_H) |_{x_H=1}}.$$

Finally we have

$$L_H(x_H) = \sum_{l=0}^{\infty} \text{Prob}[L = l] x_H^l = \sum_{l=0}^{\infty} \frac{(l + 1) \text{Prob}[a_H = l + 1]}{\frac{d}{dx_H} A_H(x_H) |_{x_H=1}} x_H^l = \frac{\frac{d}{dx_H} A_H(x_H)}{\lambda_H}.$$

### A.5. Derivation of equation (4.12)

We define  $v(x)$  as the probability distribution of the number of slots in a vacation.

$$\begin{aligned}
 V_{\pm}(z_-, z_+) &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \text{Prob}[Z_- = l, Z_+ = m] (z_-)^l (z_+)^m \\
 &= \frac{1}{\frac{d}{dz} V(z) \big|_{z=1}} \sum_{l=0}^{\infty} (z_-)^l \sum_{m=0}^{\infty} v(l+m+1) (z_+)^m \\
 &= \frac{1}{\frac{d}{dz} V(z) \big|_{z=1}} \sum_{x=0}^{\infty} \frac{1 - \left(\frac{z_-}{z_+}\right)^x}{1 - \left(\frac{z_-}{z_+}\right)} v(x) (z_+)^x \frac{1}{(z_+)} \\
 &= \frac{1}{\frac{d}{dz} V(z) \big|_{z=1}} \frac{V(z_+) - V(z_-)}{z_+ - z_-}
 \end{aligned}$$

where

$$\text{Prob}[Z_- = l, Z_+ = m] = \frac{v(l+m+1)}{\sum_{k=0}^{\infty} (k \times v(k))} = \frac{v(l+m+1)}{\frac{d}{dz} V(z) \big|_{z=1}}.$$

### A.6. Derivation of equation (4.22)

We define  $\Theta_{FB, q_H, q_N}(z)$  as the PGF for the number of slots in FB-period on the condition that the number of  $M_H$  messages is  $q_H$  and the number of  $M_N$  messages is  $q_N$  at the beginning of FB-period. This is given by

$$\Theta_{FB, q_H, q_N}(z) = (\Theta_{H,1}(z))^{q_H}.$$

By a similar argument to (4.13),  $U_{FB, N, q_H, q_N}(z, x_N)$ , the number of waiting slots of the tagged message which arrives to the system during this FB-period and other  $M_N$  messages at the end of FB-period under this condition, is given as

$$\begin{aligned}
 &U_{FB, N, q_H, q_N}(z, x_N) \\
 &= \frac{1}{\frac{d}{dz} \Theta_{FB, q_H, q_N}(z) \big|_{z=1}} \frac{\Theta_{FB, q_H, q_N}(A_N(x_N)) - \Theta_{FB, q_H, q_N}(z A_N(x_N))}{A_N(x_N) - z A_N(x_N)} L_N(x_N) \{x_N\}^{q_N} \\
 &= \frac{1}{q_H \cdot \theta_{H,1}} \frac{\{\Theta_{H,1}(A_N(x_N))\}^{q_H} \{x_N\}^{q_N} - \{\Theta_{H,1}(z A_N(x_N))\}^{q_H} \{x_N\}^{q_N}}{A_N(x_N) - z A_N(x_N)} L_N(x_N).
 \end{aligned}$$

Since probability which the tagged  $M_N$  message arrives during FB-period having begun with  $q_H$   $M_H$  messages and  $q_N$   $M_N$  messages is directly proportional to  $q_H \times \text{Prob}[q_H = l, q_N = k]$ , we have

$$\begin{aligned}
 &U_{FB, N}(z, x_N) \\
 &= \frac{\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{l \cdot \theta_{H,1}} \frac{\{\Theta_{H,1}(A_N(x_N))\}^l \{x_N\}^k - \{\Theta_{H,1}(z A_N(x_N))\}^l \{x_N\}^k}{A_N(x_N) - z A_N(x_N)} L_N(x_N) l \text{Prob}[q_H = l, q_N = k]}{\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} l \text{Prob}[q_H = l, q_N = k]} \\
 &= \frac{1}{\theta_{H,1} \chi_{FB}} \frac{X^{FB}(\Theta_{H,1}(A_N(x_N)), x_N) - X^{FB}(\Theta_{H,1}(z A_N(x_N)), x_N)}{1 - z} \frac{L_N(x_N)}{A_N(x_N)}.
 \end{aligned}$$



### A.7. Derivation of equation (4.25)

$$\begin{aligned}
C_L(z) &= \sum_{k=0}^{\infty} \text{Prob}[\text{the number of slots in C-period is } k] z^k \\
&= \sum_{k=1}^{\infty} (A_H(0)A_N(0))^{k-1} (1 - A_H(0)A_N(0)) \left(1 - \sum_{l=0}^{k-1} c_l\right) z^k + \sum_{k=0}^{\infty} (A_H(0)A_N(0))^k c_k z^k \\
&= \frac{1 - A_H(0)A_N(0)}{A_H(0)A_N(0)} \sum_{k=1}^{\infty} (A_H(0)A_N(0))^k z^k \\
&\quad - \frac{1 - A_H(0)A_N(0)}{A_H(0)A_N(0)} \sum_{k=1}^{\infty} (A_H(0)A_N(0))^k \sum_{l=0}^{k-1} c_l z^k + \sum_{k=0}^{\infty} (A_H(0)A_N(0))^k c_k z^k.
\end{aligned}$$

Here

$$\begin{aligned}
\sum_{k=1}^{\infty} (A_H(0)A_N(0))^k z^k &= \frac{zA_H(0)A_N(0)}{1 - zA_H(0)A_N(0)}, \\
\sum_{k=1}^{\infty} (A_H(0)A_N(0))^k \sum_{l=0}^{k-1} c_l z^k &= \frac{zA_H(0)A_N(0)}{1 - zA_H(0)A_N(0)} C(zA_H(0)A_N(0)), \\
\sum_{k=0}^{\infty} (A_H(0)A_N(0))^k c_k z^k &= C(zA_H(0)A_N(0)).
\end{aligned}$$

Finally we have

$$C_L(z) = \frac{(1 - A_H(0)A_N(0))z}{1 - zA_H(0)A_N(0)} (1 - C(zA_H(0)A_N(0))) + C(zA_H(0)A_N(0)).$$

### A.8. Derivation of equation (4.26)

We define  $v_l$  as the probability that the number of slots in a vacation is  $l$ . The PGF for the number of slots in a vacation along with the event no message arrives in the vacation,  $V_{\text{no arrival}}(z)$  is given by

$$\begin{aligned}
V_{\text{no arrival}}(z) &= v_1 A_H(0)A_N(0)z + v_2 (A_H(0)A_N(0))^2 z^2 + v_3 (A_H(0)A_N(0))^3 z^3 + \dots \\
&= V(zA_H(0)A_N(0)).
\end{aligned}$$

Then the PGF for the number of slots in a vacation along with the event messages arrive in the period is given by

$$V(z) - V(zA_H(0)A_N(0)).$$

Finally we have,

$$\begin{aligned}
V_L(z) &= \left(1 + V(zA_H(0)A_N(0)) + V(zA_H(0)A_N(0))^2 + \dots\right) \left(V(z) - V(zA_H(0)A_N(0))\right) \\
&= \frac{V(z) - V(zA_H(0)A_N(0))}{1 - V(zA_H(0)A_N(0))}.
\end{aligned}$$