Journal of the Operations Research
Society of Japan
Vol. 41, No. 3, September 1998

MEAN WAITING TIMES OF THE ALTERNATING TRAFFIC WITH STARTING DELAYS

Shigemichi Suzuki
Chiba Institute of Technology

Hideaki Yamashita
Tohoku University

(Received May 6, 1997; Revised March 25, 1998)

Abstract We analyze alternating traffic crossing a narrow one-lane bridge on a two-lane road. Once a car begins to cross the bridge in one direction, arriving cars from the other direction must wait, forming a queue, until all the arrivals in the first direction finish crossing the bridge. Such a situation can often be observed when road-maintenance work is carried out. Cars are assumed to arrive at the queues according to independent Poisson processes and to cross the bridge in a constant time. In addition, once cars join the queue, each car needs a starting delay, a constant, to start crossing the bridge. We model the situation where a signal controls the traffic so that the signal gives a priority to one direction at least for a fixed time. Under an assumption, the first two moments of a period during which the signal keeps giving a priority to one direction are obtained. Using a stochastic decomposition property the mean waiting times are obtained of cars to start crossing the bridge from each direction.

1. Introduction

In this paper, we analyze alternating traffic crossing a narrow one-lane bridge on a two-lane road. Once a car begins to cross the bridge in one direction, arriving cars from the other direction must wait, forming a queue, until all the arrivals in the first direction finish crossing the bridge. Such a situation can often be observed when road-maintenance work is carried out. Cars are assumed to arrive at the queues according to independent Poisson processes and to cross the bridge in a constant time. In addtion, once cars join the queue, each car needs a starting delay, a constant, to start crossing the bridge.

If the bridge is short enough to see the other side, a signal control is not necessary. The car at the head of the queue will start when it finds that there is no car of the other direction on the bridge. When there are no cars either on the bridge or in the queues, an arriving car in either direction will enter the bridge without a stop. Chatani [4] analyzed this case and obtained the mean queue lengths when all the arrivals in the other direction finish crossing the bridge, assuming the arrival rates from each side are equal.

On the other hand, if the bridge is too long (or winding) to be looked over, signals are necessary on both sides to control the traffic. We consider this case. Suppose that two sensors are set on both sides of the bridge. Consider a period during which right-hand-side traffic has a priority. If no car from the right-hand direction passes in front of the sensor during the time which a car in that direction takes to cross the bridge (there is no car in the bridge at this instant), the signal changes, giving the priority to the left-hand-side traffic. Once the signal changes, it does not change again until the left-hand side traffic disappears. Even if there is no car waiting in the left-hand side queue when the signals change, the left-hand side retains priority at least during a fixed time, called 'a forced priority time'. If any cars arrive from the left-hand side during the forced priority time, the signal is controlled by the same rule mentioned above. Otherwise the signal again changes to the right-hand side traffic when the forced priority time passes. For this model, we obtain the mean waiting times of the arriving cars to start crossing the bridge.

In the 1960s, the vehicle-actuated traffic signal control models were analyzed by Darroch, Newell and Morris [5], and Newell and Osuna [8]. Independently, the alternating priority queues were also analyzed by Avi-Itzhak, Maxwell and Miller [2], and Stidham [9]. These works became the basis of polling systems, in which a single server attends multiple queues, and many variations of the polling models have been analyzed recently . Alfa and Neuts [1] modeled platooned arrivals in road traffic using a discrete-time Markovian Arrival Process (MAP), and confirmed the intution that ignoring correlation in the arrival process results in an underestimation of the mean queue length.

The original model of our study, which incorporates neither signal control nor starting delays, was proposed and analyzed by Greenberg, Leachman and Wolff [6]. They made the approximation that cars in queue cross the bridge together in a constant time, which easily leads to the first and second moments of the lengths at embedded points and the mean delay. In this paper, we introduce starting delays and make the model more realistic.

The remainder of the paper is organized as follows. In the next section, we present the queueing model. In order to get the mean waiting times, the first and second moments of a period during which the signal keeps priority for one direction are required. In Section 3, we represent these two moments, conditioned on the numbers of waiting cars at the beginning of these periods. These conditions are removed under an assumption in Section 4. In Section 5, the mean waiting times of the cars to start crossing the bridge are expressed as the function of these two moments, using a stochastic decomposition property for the amount of work. Finally, we make concluding remarks in Section 6.

2. Model Description

The queueing model under consideration is a modified version of the simple traffic model by Greenberg, Leachman and Wolff [6], which incorporates neither the signal control nor starting delays.

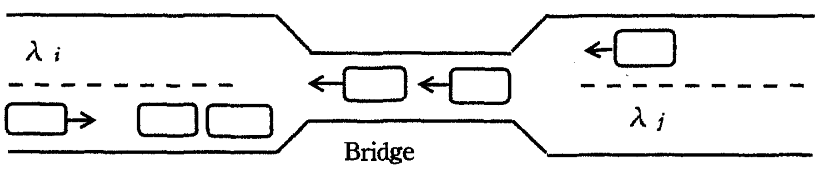

Figure 1. Alternating Traffic Model

Suppose we have a narrow one-lane bridge on a two-lane road as shown in Figure 1. Once a car begins to cross the bridge in one direction, arriving cars from the other direction must wait, forming a queue, until all the arrivals in that direction finish crossing the bridge. Queues of right-hand-side traffic and left-hand-side traffic will be referred to as Q_1 and Q_2, respectively. Cars arriving at $Q_i, i = 1, 2$, will be referred to as type-i cars. Cars arrive at the queues according to independent Poisson processes. Denote by λ_i the arrival rate at Q_i, $i = 1, 2$. Let B_i, $i = 1, 2$, be the durations of a period during which type-i cars have priority, simply called a 'type-i period'. Once type-i period ends, type-$j (j \neq i)$ cars in queue initiate type-j period. Each car in queue needs the starting delay τ seconds, that is, if there are k

cars waiting in Q_i when $B_j(j \neq i)$ ends (B_i starts at the same time), it takes $k\tau$ seconds for the kth car to start crossing the bridge. In this case, if the $(k+1)$st car arrives in $k\tau$ seconds from the starting point of B_i, then the $(k+1)$st car also needs the starting delay. Similarly, if the $(k+2)$nd car arrives in $(k+1)\tau$ seconds from the starting point of B_i, then it also needs the starting delay, and so on. Once cars start crossing the bridge, they complete crossing the bridge T seconds later, a constant, independing of the number of cars in the bridge. An arriving type-i car that finds Q_i empty but there are still some type-i cars in the bridge, can start crossing the bridge without stopping, and takes T seconds to cross the bridge. B_i will be extended as long as the next type-i car arrives while type-i cars are crossing the bridge. Even if there is no car in Q_i when B_j ends, the signal controls the traffic so that type-i cars have a priority at least V_i seconds (the forced priority time), that is, even if no type-i car arrives during V_i, type-j cars can not cross the bridge during this period. If the type-i cars arrive during V_i, then B_i will be extended in the same manner; otherwise B_i ends V_i seconds later.

In the following three sections, we analyze the queueing model and obtain the mean waiting time of the cars to start crossing the bridge for the special case $V_i = T$, $(i = 1, 2)$.

3. Conditional Duration of Type-i Period

In this section, we formulate the Laplace transform and the first two moments of B_{i,k_i}, the durations of a period during which the bridge is continuously occupied by type-i cars, conditioned that there are k_i cars waiting in Q_i when B_i starts. The conditions are removed in the next section.

3.1. The Case $k_i > 0$

First, we consider the case $k_i > 0$. As mentioned in the previous section, we have to take the starting delays into account when Q_i is not empty. Hence we divide the period B_{i,k_i} into two parts, i.e., the duration of the period where type-i cars are **waiting** in Q_i (referred to as B^W_{i,k_i}), and the duration where Q_i is empty but there are still some type-i cars **crossing** the bridge (referred to as B^C_i), that is,

$$B_{i,k} = B^W_{i,k} + B^C_i.$$

Note that the distribution of B^C_i is independent of k_i. Suppose $k_i = k > 0$. Let $N^W_{i,k}$ be the number of arriving cars during $B^W_{i,k}$, and t_1 and t_s $(s = 2, 3, \cdots, N^W_{i,k})$ be the arrival time of the 1st car from the beginning of $B^W_{i,k}$ and the inter-arrival time between the $(s-1)$st and sth cars, respectively. Since type-i cars arrive at the queues according to independent Poisson processes with rate λ_i,

$$\begin{aligned} P\{N^W_{i,k} = n\} &= \int_{t_1=0}^{k\tau} \int_{t_2=0}^{(k+1)\tau - t_1} \cdots \int_{t_{n-1}=0}^{(k+n-2)\tau - t_1 - \cdots - t_{n-2}} \int_{t_n=0}^{(k+n-1)\tau - t_1 - \cdots - t_{n-1}} \\ &\qquad \lambda_i e^{-\lambda_i t_1} \lambda_i e^{-\lambda_i t_2} \cdots \lambda_i e^{-\lambda_i t_n} e^{-\lambda_i((k+n)\tau - t_1 - \cdots - t_n)} dt_1 \cdots dt_n \\ &= \lambda_i^n e^{-\lambda_i (k+n)\tau} \int_{t_1=0}^{k\tau} dt_1 \cdots \int_{t_n=0}^{(k+n-1)\tau - t_1 - \cdots - t_{n-1}} dt_n \\ &= \lambda_i^n e^{-\lambda_i (k+n)\tau} \int_{t_1=0}^{k\tau} dt_1 \cdots \int_{t_n = t_1 + t_2 + \cdots + t_{n-1}}^{(k+n-1)\tau} dt_n \\ &= \frac{[\lambda_i (k+n)\tau]^n}{n!} \frac{k}{(k+n)} e^{-\lambda_i (k+n)\tau}, \end{aligned} \tag{3.1}$$

where the last equality can be proved using the mathematical induction. $N_{i,k}^W = n$ means $B_{i,k}^W = (k+n)\tau$; that is,

$$P\{B_{i,k}^W = (k+n)\tau\} = \frac{[\lambda_i(k+n)\tau]^n}{n!}\frac{k}{(k+n)}e^{-\lambda_i(k+n)\tau}.$$

Then the Laplace transform of $B_{i,k}^W$ is given by

$$f^*_{B_{i,k}^W}(s) = E[e^{-sB_{i,k}^W}] = \sum_{n=0}^{\infty}\frac{[\lambda_i(k+n)\tau]^{n-1}\lambda_i k\tau}{n!}e^{-(\lambda_i+s)(k+n)\tau}, \tag{3.2}$$

and the first and second moments of $B_{i,k}^W$ are obtained by

$$E(B_{i,k}^W) = -\frac{d}{ds}f^*_{B_{i,k}^W}(s)\bigg|_{s=0} = \sum_{n=0}^{\infty}\frac{[\lambda_i(k+n)\tau]^n}{n!}\frac{k}{(k+n)}(k+n)\tau e^{-\lambda_i(k+n)\tau}, \tag{3.3}$$

$$E(B_{i,k}^{W2}) = \frac{d^2}{ds^2}f^*_{B_{i,k}^W}(s)\bigg|_{s=0} = \sum_{n=0}^{\infty}\frac{[\lambda_i(k+n)\tau]^n}{n!}\frac{k}{(k+n)}(k+n)^2\tau^2 e^{-\lambda_i(k+n)\tau}. \tag{3.4}$$

Since (3.1) is still a probability distribution even when τ is replaced by τz, we have

$$\sum_{n=0}^{\infty}\frac{[\lambda_i(k+n)\tau z]^n}{n!}\frac{k}{(k+n)}e^{-\lambda_i(k+n)\tau z} = 1. \tag{3.5}$$

From the first and second derivatives of (3.5) at $z = 1$, and (3.3) and (3.4), the first and second moments of $B_{i,k}^W$ are given by

$$E(B_{i,k}^W) = \frac{k\tau}{1-\lambda_i\tau}, \tag{3.6}$$

$$E(B_{i,k}^{W2}) = \frac{k^2\tau^2}{(1-\lambda_i\tau)^2} + \frac{\lambda_i k\tau^3}{(1-\lambda_i\tau)^3}. \tag{3.7}$$

Equations (3.6) and (3.7) correspond to the first two moments of the delay cycle in $M/D/1$ queue (see sec. 1.2 in Takagi [10]), if we set the service times of the first customer and the other customers to $k\tau$ and τ, respectively.

Now we consider the Laplace transform of B_i^C. Let N_i^C be the number of arriving cars during B_i^C, and t_1 and t_s ($s = 2, 3, \cdots, N_i^C$) be the arrival time of the 1st car from the beginning of B_i^C and the inter-arrival time between the $(s-1)$st and sth cars, respectively. Given $N_i^C = n$, $B_i^C \stackrel{d}{=} T + t_1 + t_2 + \cdots + t_n$, where $t_s, s = 1, 2, \cdots, n$, are i.i.d. with a truncated exponential distribution. Hence the Laplace transform of B_i^C is given by

$$\begin{aligned} f^*_{B_i^C}(s) &= E[e^{-sB_i^C}] \\ &= \sum_{n=0}^{\infty}\int_{t_1=0}^{T}\int_{t_2=0}^{T}\cdots\int_{t_n=0}^{T}\lambda_i^n e^{-(\lambda_i+s)(T+t_1+t_2+\cdots+t_n)}dt_1 dt_2\cdots dt_n \\ &= \frac{\lambda_i+s}{se^{(\lambda_i+s)T}+\lambda_i}, \end{aligned} \tag{3.8}$$

and the first two moments of B_i^C is obtained by

$$E(B_i^C) = -\frac{d}{ds}f^*_{B_i^C}(s)\bigg|_{s=0} = \frac{e^{\lambda_i T}-1}{\lambda_i}, \tag{3.9}$$

$$E(B_i^{C2}) = \frac{d^2}{ds^2}f^*_{B_i^C}(s)\bigg|_{s=0} = \frac{2e^{\lambda_i T}(e^{\lambda_i T}-\lambda_i T-1)}{\lambda_i^2}, \tag{3.10}$$

which coincide with the results in Greenberg, Leachman and Wolff [6]. Since $B^W_{i,k}$ and B^C_i are independent each other, we have the Laplace transform of $B_{i,k}$ and the first two moments of duration of $B_{i,k}$ using (3.2) and (3.6) $\sim$ (3.10):

$$\begin{aligned} f^*_{B_{i,k}}(s) &= f^*_{B^W_{i,k}}(s) f^*_{B^C_i}(s) \\ &= \frac{\lambda_i + s}{se^{(\lambda_i+s)T} + \lambda_i} \sum_{n=0}^{\infty} \frac{[\lambda_i(k+n)\tau]^{n-1}\lambda_i k\tau}{n!} e^{-(\lambda_i+s)(k+n)\tau}, \qquad (3.11) \\ E(B_{i,k}) &= \frac{k\tau}{1-\lambda_i\tau} + \frac{e^{\lambda_i T}-1}{\lambda_i}, \qquad (3.12) \\ E(B^2_{i,k}) &= \frac{\tau^2}{(1-\lambda_i\tau)^2} k(k-1) + \frac{\tau^2}{(1-\lambda_i\tau)^3} k + \frac{2\tau(e^{\lambda_i T}-1)}{(1-\lambda_i\tau)\lambda_i} k \\ &\quad + \frac{2e^{\lambda_i T}(e^{\lambda_i T} - \lambda_i T - 1)}{\lambda_i^2}. \qquad (3.13) \end{aligned}$$

3.2. The Case $k_i = 0$

Next we consider the case $k_i = 0$, where there is no type-i cars waiting in Q_i when B_j ends (B_i starts). We refer to the duration as $B_{i,0}$. In this case, a signal controls the traffic, so that type-i cars have priority for at least V_i seconds. That is, $B_{i,0}$ continues for at least V_i seconds. If no type-i car arrives during V_i, an event with probability $e^{-\lambda_i V_i}$, duration of $B_{i,0} = V_i$, and hence

$$P\{B_{i,0} = V_i\} = e^{-\lambda_i V_i}.$$

On the other hand, if any type-i cars arrive during V_i, $B_{i,0}$ will be extended. The distribution of the extended part is the same as B^C_i. Here, $B^E_{i,0}$ denotes the duration from the starting instant of $B_{i,0}$ to the instant when the first type-i car arrives at Q_i, i.e., the period during which the bridge is **empty**. The Laplace transform of $B^E_{i,0}$ is easily obtained by

$$f^*_{B^E_{i,0}}(s) = \frac{1}{1-e^{-\lambda_i V_i}} \int_0^{V_i} \lambda_i e^{-(\lambda_i+s)t} dt = \frac{1}{1-e^{-\lambda_i V_i}} \frac{\lambda_i}{\lambda_i + s} [1 - e^{-(\lambda_i+s)V_i}].$$

Combining two cases, we have the Laplace transform of $B_{i,0}$ as follows:

$$f^*_{B_{i,0}}(s) = e^{-\lambda_i V_i} e^{-sV_i} + f^*_{B^E_{i,0}}(s) f^*_{B^C_i}(s) = e^{-(\lambda_i+s)V_i} + (1 - e^{-\lambda_i V_i}) \frac{\lambda_i[1-e^{-(\lambda_i+s)V_i}]}{\lambda_i + se^{(\lambda_i+s)T}}. \qquad (3.14)$$

Accordingly, the first and second moments of $B_{i,0}$ are

$$\begin{aligned} E(B_{i,0}) &= -\frac{d}{ds} f^*_{B_{i,0}}(s)\Big|_{s=0} = \frac{(1-e^{-\lambda_i V_i})e^{\lambda_i T}}{\lambda_i}, \qquad (3.15) \\ E(B^2_{i,0}) &= \frac{d^2}{ds^2} f^*_{B_{i,0}}(s)\Big|_{s=0} = 2e^{\lambda_i T}\left[\frac{e^{\lambda_i T} - \lambda_i T - e^{\lambda_i(T-V_i)}}{\lambda_i^2} + \frac{(T-V_i)e^{-\lambda_i V_i}}{\lambda_i}\right]. \qquad (3.16) \end{aligned}$$

4. Moments of Type-i Period

In this section, we obtain the first two moments of a type-i period, $E(B_i)$ and $E(B_i^2)$, when the forced priority time $V_i = T$, the crossing time of the bridge, using the formulation of the Laplace transform of B_{i,k_i} obtained in the previous section.

First we consider an embedded Markov chain embedded at the instants (switch points) when B_j ends (B_i starts at the same time). Let $i (i = 1, 2)$ be the indicator variable that shows which duration (B_1 or B_2) starts. k_i denotes the number of cars waiting in Q_i when B_i starts. Let X_n be the state at (just after) the nth switch point. Then $\{X_n\}$ has the Markov property. Since B_1 and B_2 appear exactly alternately, the Markov chain is periodic with period 2. If we observe the Markov chain at every 2 embedded points, it comes down to the aperiodic Markov chain, so it may have the steady state probabilities under some stability conditions. Here, we assume that such modified the Markov chain has the steady state probabilities denoted by p_{i,k_i} for the state (i, k_i). Half the time the chain is in $(1, k_1)$ states, and half the time it is in $(2, k_2)$ states; that is $\sum_{k_i=0}^{\infty} p_{i,k_i} = 1/2$ for $i = 1, 2$. Let q_{i,k_i,k_j} denotes the transition probability from state (i, k_i) to (j, k_j) $(i \neq j)$. Then we have the system of linear equations as follows:

$$\begin{pmatrix} p_{1,0} \\ p_{1,1} \\ p_{1,2} \\ \vdots \\ \hline p_{2,0} \\ p_{2,1} \\ p_{2,2} \\ \vdots \end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & & q_{2,0,0} & q_{2,1,0} & q_{2,2,0} & \cdots \\ & 0 & & & q_{2,0,1} & q_{2,1,1} & q_{2,2,1} & \cdots \\ & & & & q_{2,0,2} & q_{2,1,2} & q_{2,2,2} & \cdots \\ & & & & \vdots & \vdots & \vdots & \ddots \\ \hline q_{1,0,0} & q_{1,1,0} & q_{1,2,0} & \cdots & & & & \\ q_{1,0,1} & q_{1,1,1} & q_{1,2,1} & \cdots & & & & \\ q_{1,0,2} & q_{1,1,2} & q_{1,2,2} & \cdots & & 0 & & \\ \vdots & \vdots & \vdots & \ddots & & & & \end{array}\right) \begin{pmatrix} p_{1,0} \\ p_{1,1} \\ p_{1,2} \\ \vdots \\ \hline p_{2,0} \\ p_{2,1} \\ p_{2,2} \\ \vdots \end{pmatrix}$$

Now, we obtain the transition probabilities, q_{i,k_i,k_j}. q_{i,k_i,k_j} can be expressed by

$$q_{i,k_i,k_j} = \int_0^\infty \frac{(\lambda_j t)^{k_j}}{k_j!} e^{-\lambda_j t} f_{B_{i,k_i}}(t)dt$$

where $i, j = 1, 2, i \neq j$, and $f_{B_{i,k_i}}(t)$ is the probability density function of B_{i,k_i} when $k_i \neq 0$, and the integral should be understood as Laplace-Styelties transform when $k_i = 0$. On the other hand,

$$f_{B^*_{i,k_i}}(s) = \int_0^\infty f_{B_{i,k_i}}(t)e^{-st}dt,$$

which leads

$$(-1)^{k_j}\lambda_j^{k_j}\frac{d^{k_j}}{ds^{k_j}}f^*_{B_{i,k_i}}(s) = \int_0^\infty (\lambda_j t)^{k_j} f_{B_{i,k_i}}(t)e^{-st}dt.$$

Accordingly, we have the following relation:

$$q_{i,k_i,k_j} = \frac{(-1)^{k_j}\lambda_j^{k_j}}{k_j!}\frac{d^{k_j}}{ds^{k_j}}f^*_{B_{i,k_i}}(s)\bigg|_{s=\lambda_j} \tag{4.1}$$

Therefore, we are able to calculate the transition probabilities using the Laplace transform of $f^*_{B_{i,k_i}}(s)$ in (3.11) and (3.14). In particular,

$$q_{i,0,k_j} = e^{-\lambda_i V_i}\frac{(\lambda_j V_i)^{k_j}}{k_j!}e^{-\lambda_j V_i} + \frac{(-1)^{k_j}\lambda_j^{k_j}}{k_j!}\frac{d^{k_j}}{ds^{k_j}}\frac{\lambda_i[1-e^{-(\lambda_i+s)V_i}]}{\lambda_i + se^{(\lambda_i+s)T}}\bigg|_{s=\lambda_j} \tag{4.2}$$

Here, we derive the sufficient conditions of the modified Markov chain to be positive recurrent. Now, we introduce the statements by Karlin [7].

Lemma *Suppose a Markov chain is irreducible. Then the sufficient condition for the Markov* *chain to be recurrent is that there exits the sequence $\{y_u\}$ which satisfies the following conditions:*

$$\sum_{v=0}^{\infty} r_{u,v} y_v \le y_u \quad \text{for } u \neq 0$$

and

$$y_u \to \infty \quad \text{as} \quad u \to \infty,$$

where $r_{u,v}$ is the transition probability of the Markov chain.
Let us set the sequence $\{y_u\}$ in above lemma as

$$y_u = \begin{cases} u & \text{if } u > 0, \\ -M & \text{if } u = 0, \end{cases}$$

where M is a positive number, and consider m such that

$$\forall u \ge m, \quad \sum_{v=0}^{\infty} r_{u,v} v \le u. \tag{4.3}$$

Since $\sum_{v=1}^{\infty} r_{u,v} v$ is finite and $r_{u,0}$ is positive for every u in this problem, we have

$$\sum_{v=0}^{\infty} r_{u,v} y_v = \sum_{v=1}^{\infty} r_{u,v} v - M r_{u,0} \le u,$$

for every u, and then these $\{y_u\}$ satisfy the above conditions for sufficient large M. Therefore we may derive the conditions for existence of m which satisfies (4.3).

Here, from (4.1) and (3.11), for $k_i > 0$ we get

$$\begin{aligned} \sum_{k_j=0}^{\infty} k_j q_{i,k_i,k_j} &= -\lambda_j \sum_{k_j=1}^{\infty} \frac{(-\lambda_j)^{k_j-1}}{(k_j-1)!} \frac{d^{k_j-1}}{ds^{k_j-1}} \frac{d}{ds} f^*_{B_{i,k_i}}(s) \bigg|_{s=\lambda_j} \\ &= -\lambda_j \frac{d}{ds} f^*_{B_{i,k_i}}(s-\lambda_j) \bigg|_{s=\lambda_j} = \lambda_j \left[\frac{k_i \tau}{1-\lambda_i \tau} + \frac{e^{\lambda_i T}-1}{\lambda_i} \right]. \end{aligned} \tag{4.4}$$

Using (4.4),

$$\begin{aligned} \sum_{k_i'=0}^{\infty} r_{k_i,k_i'} y_{k_i'} &= \sum_{k_j=0}^{\infty} \sum_{k_i'=0}^{\infty} q_{i,k_i,k_j} q_{j,k_j,k_i'} k_i' \\ &= \sum_{k_j=0}^{\infty} q_{i,k_i,k_j} \lambda_i \left[\frac{k_j \tau}{1-\lambda_j \tau} + \frac{e^{\lambda_j T}-1}{\lambda_j} \right] \\ &= \frac{(\lambda_i \tau)(\lambda_j \tau)}{(1-\lambda_i \tau)(1-\lambda_j \tau)} k_i + \frac{\lambda_j \tau (e^{\lambda_i T}-1)}{1-\lambda_j \tau} + \frac{\lambda_i (e^{\lambda_j T}-1)}{\lambda_j}. \end{aligned} \tag{4.5}$$

Then, (4.3) is equivalent to

$$\frac{1-\lambda_i \tau - \lambda_j \tau}{(1-\lambda_i \tau)(1-\lambda_j \tau)} k_i \ge \frac{\lambda_j \tau (e^{\lambda_i T}-1)}{1-\lambda_j \tau} + \frac{\lambda_i (e^{\lambda_j T}-1)}{\lambda_j}.$$

If

$$1 - \lambda_j \tau - \lambda_i \tau > 0, \tag{4.6}$$

then we can determine m such that m is the minimum integer which satisfies

$$m \geq \frac{\lambda_j\tau(1-\lambda_i\tau)(e^{\lambda_i T}-1)}{1-\lambda_i\tau-\lambda_j\tau} + \frac{\lambda_i(1-\lambda_i\tau)(1-\lambda_j\tau)(e^{\lambda_j T}-1)}{\lambda_j(1-\lambda_i\tau-\lambda_j\tau)}.$$

Hence (4.6) is the sufficient condition for the modified Markov chain to be recurrent. We will now show (4.6) is a sufficient condition for the Markov chain to be **positive** recurrent. In order to establish this, we will show that starting from some state (i, k_i) after an infinite number of type-i periods and type-j periods the expected number of cars waiting in Q_i when B_i starts converges to some positive value.

Assume (4.6) hereafter. Let $k_i^{(n)}$ (or $k_j^{(n)}$) be the number of cars waiting at Q_i (or Q_j) at the beginning of the n-th B_i (or B_j) period starting from $(i, k_i^{(0)})$ initially. Then the expected number of $k_i^{(n)}$ can be evaluated using (4.5) n times as follows:

$$\begin{aligned}
&\sum_{k_j^{(1)}=0}^{\infty}\sum_{k_i^{(1)}=0}^{\infty}\cdots\sum_{k_j^{(n)}=0}^{\infty}\sum_{k_i^{(n)}=0}^{\infty} q_{i,k_i^{(0)},k_j^{(1)}}q_{j,k_j^{(1)},k_i^{(1)}}\cdots q_{i,k_i^{(n-1)},k_j^{(n)}}q_{j,k_j^{(n)},k_i^{(n)}}k_i^{(n)} \\
&= \sum_{k_j^{(1)}=0}^{\infty}\cdots\sum_{k_i^{(n-1)}=0}^{\infty} q_{i,k_i^{(0)},k_j^{(1)}}\cdots q_{j,k_j^{(n-1)},k_i^{(n-1)}}[s_is_jk_i^{(n-1)}+s_it_j+t_i] \\
&= \cdots \\
&= (s_is_j)^n k_i^{(0)} + (s_it_j+t_i)[(s_is_j)^{n-1}+(s_is_j)^{n-2}+\cdots+1],
\end{aligned}$$

where

$$s_i = \frac{\lambda_i\tau}{1-\lambda_j\tau}, \quad t_i = \frac{\lambda_i(e^{\lambda_j T}-1)}{\lambda_j},$$

$$s_j = \frac{\lambda_j\tau}{1-\lambda_i\tau}, \quad t_j = \frac{\lambda_j(e^{\lambda_i T}-1)}{\lambda_i}.$$

Since $0 < s_is_j < 1$ because of (4.6),

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k_j^{(1)}=0}^{\infty}\sum_{k_i^{(1)}=0}^{\infty}\cdots\sum_{k_j^{(n)}=0}^{\infty}\sum_{k_i^{(n)}=0}^{\infty} q_{i,k_i^{(0)},k_j^{(1)}}q_{j,k_j^{(1)},k_i^{(1)}}\cdots q_{i,k_i^{(n-1)},k_j^{(n)}}q_{j,k_j^{(n)},k_i^{(n)}}k_i^{(n)} \\
&= \frac{s_it_j+t_i}{1-s_is_j} \\
&= [\frac{\lambda_i\tau}{1-\lambda_j\tau}\frac{\lambda_j(e^{\lambda_i T}-1)}{\lambda_i} + \frac{\lambda_i(e^{\lambda_j T}-1)}{\lambda_j}][\frac{(1-\lambda_i\tau)(1-\lambda_j\tau)}{1-\lambda_i\tau-\lambda_j\tau}]. \qquad (4.7)
\end{aligned}$$

Then the left hand side of (4.7) is finite. If the Markov chain is recurrent null, the left hand side of (4.7) does not converge. Hence if (4.6) holds, then the Markov chain can be proved to be positive recurrent.

Note that (4.6) means that the number of arriving cars from the both hand sides during the starting delay τ must be less than 1. Throughout the paper, the stability condition (4.6) is assumed to hold.

Now, we are ready to obtain the first two moments of B_i. First, using the steady state equations

$$p_{j,k_j} = \sum_{k_i=0}^{\infty} p_{i,k_i}q_{i,k_i,k_j} \quad (i,j=1,2, i\neq j),$$

we have

$$\sum_{k_j=0}^{\infty} k_j p_{j,k_j} = \sum_{k_j=0}^{\infty}\sum_{k_i=0}^{\infty} k_j p_{i,k_i} q_{i,k_i,k_j} = \sum_{k_i=0}^{\infty}\sum_{k_j=0}^{\infty} k_j q_{i,k_i,k_j} p_{i,k_i}. \tag{4.8}$$

From (4.2), we have

$$\begin{aligned}
\sum_{k_j=0}^{\infty} k_j q_{i,0,k_j} &= \sum_{k_j=0}^{\infty} k_j e^{-\lambda_i V_i}\frac{(\lambda_j V_i)^{k_j}}{k_j!}e^{-\lambda_j V_i} \\
&\quad + \lambda_j \sum_{k_j=1}^{\infty} \frac{(-\lambda_j)^{k_j-1}}{(k_j-1)!}\frac{d^{k_j-1}}{ds^{k_j-1}}\frac{d}{ds}\frac{\lambda_i[1-e^{-(\lambda_i+s)V_i}]}{\lambda_i + se^{(\lambda_i+s)T}}\Bigg|_{s=\lambda_j} \\
&= \lambda_j V_i e^{-\lambda_i V_i} - \lambda_j \frac{d}{ds}\frac{\lambda_i[1-e^{-(\lambda_i-\lambda_j+s)V_i}]}{\lambda_i+(s-\lambda_j)e^{(\lambda_i-\lambda_j+s)T}}\Bigg|_{s=\lambda_j} \\
&= \frac{\lambda_j}{\lambda_i}(1-e^{-\lambda_i V_i})e^{\lambda_i T} \\
&= \lambda_j\frac{e^{\lambda_i T}-1}{\lambda_i} - \lambda_j\frac{e^{\lambda_i(T-V_i)}-1}{\lambda_i}.
\end{aligned} \tag{4.9}$$

Substituting (4.4) and (4.9) into (4.8), we get

$$\begin{aligned}
\sum_{k_j=0}^{\infty} k_j p_{j,k_j} &= \sum_{k_i=0}^{\infty}\left[\frac{\lambda_j\tau}{1-\lambda_i\tau}k_i + \frac{\lambda_j(e^{\lambda_i T}-1)}{\lambda_i}\right]p_{i,k_i} - \lambda_j\frac{e^{\lambda_i(T-V_i)}-1}{\lambda_i}p_{i,0} \\
&= \frac{\lambda_j\tau}{1-\lambda_i\tau}\sum_{k_i=0}^{\infty} k_i p_{i,k_i} + \frac{\lambda_j(e^{\lambda_i T}-1)}{2\lambda_i} - \lambda_j\frac{e^{\lambda_i(T-V_i)}-1}{\lambda_i}p_{i,0},
\end{aligned} \tag{4.10}$$

where we use $\sum_{k_i=0}^{\infty} p_{i,k_i} = 1/2$ in the last equality. In the same manner, using

$$\begin{aligned}
\sum_{k_j=0}^{\infty} k_j(k_j-1)q_{i,k_i,k_j} &= \frac{\lambda_j^2\tau^2}{(1-\lambda_i\tau)^2}k(k-1) \\
&\quad +\lambda_j^2\left[\frac{\tau^2}{(1-\lambda_i\tau)^3} + \frac{2\tau(e^{\lambda_i T}-1)}{(1-\lambda_i\tau)\lambda_i}\right]k + \frac{2\lambda_j^2 e^{\lambda_i T}(e^{\lambda_i T}-\lambda_i T-1)}{\lambda_i^2},
\end{aligned}$$

$$\sum_{k_j=0}^{\infty} k_j(k_j-1)q_{i,0,k_j} = \frac{2\lambda_j^2 e^{\lambda_i T}}{\lambda_i^2}\{e^{\lambda_i T} - \lambda_i T - e^{\lambda_i(T-V_i)} + \lambda_i(T-V_i)e^{-\lambda_i V_i}\},$$

we get

$$\begin{aligned}
&\sum_{k_j=0}^{\infty} k_j(k_j-1)p_{j,k_j} \\
&= \frac{\lambda_j^2\tau^2}{(1-\lambda_i\tau)^2}\sum_{k_i=0}^{\infty} k_i(k_i-1)p_{i,k_i} + \lambda_j^2\left[\frac{\tau^2}{(1-\lambda_i\tau)^3} + \frac{2\tau(e^{\lambda_i T}-1)}{(1-\lambda_i\tau)\lambda_i}\right]\sum_{k_i=0}^{\infty} k_i p_{i,k_i} \\
&\quad + \frac{\lambda_j^2 e^{\lambda_i T}(e^{\lambda_i T}-\lambda_i T-1)}{\lambda_i^2} + \frac{2\lambda_j^2 e^{\lambda_i T}}{\lambda_i^2}\{\lambda_i(T-V_i)e^{-\lambda_i V_i} + 1 - e^{\lambda_i(T-V_i)}\}p_{i,0}.
\end{aligned} \tag{4.11}$$

In order to get the left hand sides of (4.10) and (4.11), $p_{1,0}$ and $p_{2,0}$ are neccessary to be expressed by the known parameters, which seems to be difficult.

Here, we consider the special case $V_i = T$, $(i = 1, 2)$. That is, the signal keeps the priority for one direction during the time a car needs to cross the bridge, even if no car in that direction is waiting when the signal changes. This assumption seems to be natural because in this case, the signal changes when no cars passes in front of sensor during time T. When $V_i = T$, (4.10) and (4.11) are equivalent to

$$\sum_{k_j=0}^{\infty} k_j p_{j,k_j} = \frac{\lambda_j \tau}{1-\lambda_i \tau} \sum_{k_i=0}^{\infty} k_i p_{i,k_i} + \frac{\lambda_j (e^{\lambda_i T} - 1)}{2\lambda_i}, \quad (i,j = 1,2, i \neq j),$$

and

$$\begin{aligned}
&\sum_{k_j=0}^{\infty} k_j(k_j-1) p_{j,k_j} \\
&= \frac{\lambda_j^2 \tau^2}{(1-\lambda_i\tau)^2} \sum_{k_i=0}^{\infty} k_i(k_i-1) p_{i,k_i} + \lambda_j^2 \left[\frac{\tau^2}{(1-\lambda_i\tau)^3} + \frac{2\tau(e^{\lambda_i T}-1)}{(1-\lambda_i\tau)\lambda_i} \right] \sum_{k_i=0}^{\infty} k_i p_{i,k_i} \\
&+ \frac{\lambda_j^2 e^{\lambda_i T}(e^{\lambda_i T} - \lambda_i T - 1)}{\lambda_i^2}, \quad (i,j = 1,2, i \neq j).
\end{aligned}$$

By solving the above systems of linear equations, we finally get

$$\sum_{k_i=0}^{\infty} k_i p_{i,k_i} = \frac{(1-\lambda_i\tau)\{\lambda_j^2\tau(e^{\lambda_i T}-1) + \lambda_i(1-\lambda_j\tau)(e^{\lambda_j T}-1)\}}{2(1-\lambda_i\tau-\lambda_j\tau)\lambda_j}, \tag{4.12}$$

$$\begin{aligned}
\sum_{k_i=0}^{\infty} k_i(k_i-1) p_{i,k_i} &= \frac{(1-\lambda_i\tau)^2(1-\lambda_j\tau)^2}{(1-\lambda_i^2\tau^2-\lambda_j^2\tau^2)} \\
&\times \left[\frac{\lambda_i^2\lambda_j\tau^2}{(1-\lambda_j\tau)^2} \frac{\lambda_j^2\tau(e^{\lambda_i T}-1) + \lambda_i(1-\lambda_j\tau)(e^{\lambda_j T}-1)}{2(1-\lambda_i\tau-\lambda_j\tau)} \left\{ \frac{\tau^2}{(1-\lambda_i\tau)^2} + \frac{2\tau(e^{\lambda_i T}-1)}{\lambda_i} \right\} \right. \\
&+ \lambda_i \frac{\lambda_i^2\tau(e^{\lambda_j T}-1) + \lambda_j(1-\lambda_i\tau)(e^{\lambda_i T}-1)}{2(1-\lambda_i\tau-\lambda_j\tau)} \left\{ \frac{\tau^2}{(1-\lambda_j\tau)^2} + \frac{2\tau(e^{\lambda_j T}-1)}{\lambda_j} \right\} \\
&\left. + \frac{\lambda_j^2\tau^2 e^{\lambda_i T}(e^{\lambda_i T}-\lambda_i T-1)}{(1-\lambda_j\tau)^2} + \frac{\lambda_i^2 e^{\lambda_j T}(e^{\lambda_j T}-\lambda_j T-1)}{\lambda_j^2} \right].
\end{aligned} \tag{4.13}$$

On the other hand, from (3.12),(3.13),(3.15),(3.16), and when $V_i = T$, the first two moments of type-i period are expressed as follows:

$$E(B_i) = 2\sum_{k_i=0}^{\infty} p_{i,k_i} E(B_{i,k_i}) = \frac{2\tau}{1-\lambda_i\tau} \sum_{k_i=0}^{\infty} k_i p_{i,k_i} + \frac{(e^{\lambda_i T}-1)}{\lambda_i}, \tag{4.14}$$

$$\begin{aligned}
E(B_i^2) &= 2\sum_{k_i=0}^{\infty} p_{i,k_i} E(B_{i,k_i}^2) \\
&= \frac{2\tau^2}{(1-\lambda_i\tau)^2} \sum_{k_i=0}^{\infty} k_i(k_i-1) p_{i,k_i} \\
&+ 2\left[\frac{\tau^2}{(1-\lambda_i\tau)^3} + \frac{2\tau(e^{\lambda_i T}-1)}{(1-\lambda_i\tau)\lambda_i} \right] \sum_{k_i=0}^{\infty} k_i p_{i,k_i} + \frac{2e^{\lambda_i T}(e^{\lambda_i T}-\lambda_i T-1)}{\lambda_i^2}.
\end{aligned} \tag{4.15}$$

As a result, substituting (4.12) and (4.13) into (4.14) and (4.15), the first two moments of duration of B_i can be obtained. In particular,

$$E(B_i) = \frac{\tau\{\lambda_j^2\tau(e^{\lambda_i T}-1) + \lambda_i(1-\lambda_j\tau)(e^{\lambda_j T}-1)\}}{(1-\lambda_i\tau-\lambda_j\tau)\lambda_j} + \frac{e^{\lambda_i T}-1}{\lambda_i}.$$

5. Mean Waiting Time

In this section, the mean waiting time of the arriving cars to start crossing the bridge is expressed as the function of $E(B_i)$ and $E(B_i^2)$, consequently, when $V_i = T$ $(i = 1, 2)$, the mean waiting time can be calculated using the results for $E(B_i)$ and $E(B_i^2)$ obtained in Section 4.

Let us consider the waiting time of a random (tagged) type-i car. In the traffic model, the waiting times of the type-i cars (a tagged car) consist of two elements. One is the time until B_i starts (If the tagged car arrives at Q_i during B_j, $j \neq i$, this element is not zero.), and the other is the amount of starting delays of the type-i cars in front of the tagged car, including the starting delay of the tagged car. The former is equivalent to the forward reccurence time of B_j, with expectation $E(B_j^2)/2E(B_j)$, and is experienced by these cars with probability $E(B_j)/(E(B_i) + E(B_j))$. Thus, this part of the mean waiting time is equal to $E(B_j^2)/2(E(B_i) + E(B_j))$.

Now, we consider the second element, the amount of starting delays of the type-i cars in front of the tagged car. As mentioned in section 3, B_i consists of two part, i.e., the period during which type-i cars are **waiting**, and the period during which Q_i is empty but there are still some type-i cars **crossing** the bridge. The arriving cars during the waiting period have starting delays, but ones during crossing period can enter the bridge without a stop. Note that if there is no car waiting when B_i starts, that B_i is totally included in the crossing period. If we remove the crossing periods of B_i from the process, the remaining process consists of the waiting periods of B_i and the waiting and crossing periods of B_j, which can be regarded, for the type-i cars, as the process of an $M/G/1$ vacation system with the exhaustive service, with the arrival rate is λ_i, service times are τ, and vacation periods with first and second moments, $E(B_j)$ and $E(B_j^2)$. Here we utilize the stochastic decomposition property proved by Boxma and Groenendijk [3]. That is, the amount of work in the vacation system at an arbitrary epoch, U, is distributed as the sum of the amount of work in the 'corresponding' (without vacations) $M/G/1$ system at an arbitrary epoch, V, and the amount of work in the vacation system at an arbitrary epoch in a vacation interval, Y. Consequently, we have

$$E(U) = E(V) + E(Y).$$

Since the backward reccurence time of the vacation period for type-i cars is equal to $E(B_j^2)/2E(B_j)$, and the type-i cars arrive at Q_i according to an independent Poisson process with rate λ_i, we get

$$E(Y) = \frac{\lambda_i \tau E(B_j^2)}{2E(B_j)}.$$

$E(V)$ can be easily obtained by

$$E(V) = \frac{\lambda_i \tau^2}{2(1 - \lambda_i \tau)},$$

and then we have

$$E(U) = \frac{\lambda_i \tau E(B_j^2)}{2E(B_j)} + \frac{\lambda_i \tau^2}{2(1 - \lambda_i \tau)}.$$

As Poisson arrivals see time averages, the expected starting delay which the tagged car experiences is the sum of $E(U)$ and τ, the starting delay of the tagged car itself, if the tagged car arrives at Q_i during the waiting periods of B_i or type-j periods. However, the tagged car arrives during the crossing periods of B_i, then it does not wait to enter the bridge. That probability is $E(B_i^C)/(E(B_i)+E(B_j))$ when $V_i = T$, because $E(B_{i,0})$ is equal to $E(B_i^C)$ in the case of $V_i = T$. Hence, the latter part of the mean waiting time is equal to $(E(U)+\tau)\{1 - E(B_i^C)/(E(B_i)+E(B_j))\}$.

From the above discussion, the mean waiting time of type-i cars, $E(W_i)$, can be expressed by

$$\begin{aligned} E(W_i) &= \frac{E(B_j^2)}{2(E(B_i)+E(B_j))} + (E(U)+\tau)\left[1 - \frac{E(B_i^C)}{E(B_i)+E(B_j)}\right] \qquad (5.1) \\ &= \frac{E(B_j^2)}{2(E(B_i)+E(B_j))} + \left[\frac{\tau E(B_j^2)}{2E(B_j)} + \frac{\tau^2}{2(1-\lambda_i\tau)} + \tau\right]\left[\lambda_i - \frac{e^{\lambda_i T}-1}{E(B_i)+E(B_j)}\right]. \end{aligned}$$

Then, we can obtain the mean waiting time by substituting (4.14) and (4.15) into (5.1) when $V_i = T$.

6. Concluding Remarks

In this paper, we modeled the alternating traffic crossing a narrow one-lane bridge on a two-lane road in whcih the signal controls priorities. We obtained the closed forms for the first and second moments of type-i periods ($i = 1, 2$) in the special case $V_i = T$. This case is resonable since it corresponds to the situation where the signal changes when no cars passes in front of sensor during the time it takes cars to cross the bridge. Then, we expressed the mean waiting time of the cars to start crossing the bridge as the function of these two moments using the stochastic decomposition property.

The methodology in this paper may be extended to get the higher moments of the waiting time of the cars to start crossing the bridge, that is, the nth moment of the waiting time can be obtained using the $(n+1)$st and lower moments of type-i periods ($i = 1, 2$).

Acknowledgments

The authors would like to thank Dr. Ronald Wolff for careful reading of an earlier version of this paper and for many helpful comments. The authors also would like to thank two referees for their helpful comments and suggestions.

References

[1] A.S.Alfa and M.F.Neuts: Modelling vehicular traffic using the discrete time Markovian arrival process. *Transportation Science,* **29** (1995) 109-117.

[2] B.Avi-Itzhak, W.L.Maxwell, and L.W.Miller: Queueing with alternating priorities. *Operations Research,* **13** (1965) 306-318.

[3] O.J.Boxma and W.P.Groenendijk: Pseudo-conservation laws in cyclic-service systems. *Journal of Applied Probability,* **24** (1987) 949-964.

[4] K.Chatani: *Modeling and analysis of the alternating traffic* (Master Thesis in Dept. of Mechanical Eng., Sophia University, 1993).

[5] J.N.Darroch, G.F.Newell, and R.W.Morris: Queues for a vehicle-actuated traffic light. *Operations Research,* **12** (1964) 882-895.

[6] B.S.Greenberg, R.C.Leachman, and R.W.Wolff: Predicting dispatching delays on a low speed single track railroad. *Transportation Science,* **22** (1988) 31-38.

[7] S.Karlin: *A first course in stochastic processes* (Academic Press, New York and London, 1971).

[8] G.F.Newell and E.E.Osuna: Properties of vehicle-actuated signals: II. Two way streets. *Transportation Science,* **13** (1969) 99-125.

[9] S.Stidham: Regenerative processes in the theory of queues, with applications to the alternating-priority queue. *Advances in Applied Probability,* **4** (1972) 542-577.

[10] H.Takagi: *Queueing Analysis Vol.1* (North-Holland, Amsterdam, 1991).

Hideaki Yamashita
Faculty of Economics
Tohoku University
Aoba-ku, Sendai, 980-8576, Japan.
hideaki@econ.tohoku.ac.jp