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AN EXPLICIT SOLUTION FOR AN M/GI/1/N QUEUE WITH VACATION TIME AND EXHAUSTIVE SERVICE DISCIPLINE *

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Abstract We consider an M/GI/1/N queue with vacation time and exhaustive service discipline. We show that the embedded Markov chain formed by the customer departure epochs makes the analysis simple and it enables us to find an explicit solution for the server-vacation queue. We also find an explicit solution for the ordinary (non-vacation) queue.

1. Introduction

The time division multiple access (TDMA) scheme is practical in the areas of communications. Communication engineers frequently encounter a teletraffic issue how to design a buffer capacity in the TDMA environment (see Stuck and Arthurs [17]). The issue then necessiates a single-server finite capacity queue with *vacation time* and *exhaustive* service discipline.

By *vacation time*, we mean that the server becomes unavailable for occasional intervals of time, and by *exhaustive* we mean that customers are served continuously until there is no customer in the system (see Doshi [5], and Takagi [18]). The vacation time corresponds to a constant slotted time period in the TDMA system.

Under the exhaustive vacation policy, the well known stochastic decomposition formula (see Doshi [15], Fuhrmann and Cooper [8], Kroese and Schmidt [10] and Miyazawa [15]) will be useful for an infinite capacity queue. However, if we would like to evaluate the loss probability we have to treat a finite capacity system rather than infinite capacity systems.

Assuming Poisson input and a finite capacity queue Lee [11, 12] already provided a numerical algorithm for this system via the standard embedded Markov chain technique. As the embedded points, he took the service completion epochs and server vacation completion epochs. To obtain the queue length distribution at an arbitrary time, he applied the supplementary variable technique and the sample biasing technique.

Here, we treat the same queueing system as in Lee [11] but present a simpler analysis than Lee's. We show that service completion epochs are enough for the queue length to form an embedded Markov chain (server vacation completion epochs are redundant). This simpler analysis enables us to find an explicit solution for the steady-state queue length distribution. To the best of the authors' knowledge, there are no results on the explicit solution for finite capacity queues. A main contribution of the paper is then to provide the explicit solution. Combining a couple of simple qualitative results, we straightforwardly

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obtain the queue length distribution at an arbitrary time from which engineers can evaluate their most interesting performance measures.

This paper is organized as follows. Section 2 describes the queueing model and introduces the notation. In Section 3, we derive a recursive scheme for the steady-state queue-length distribution at departure epochs. Note that as embedded Markov chain, we use only the departure epochs (as opposed to both departure and vacation completion epochs by Lee [11]), and this simplification leads to an explicit solution for the queue length distribution at departure epochs. Section 4 gives a probabilistic argument to obtain the steady-state queue length distribution at an arbitrary epoch. In the last seciton we give some concluding remarks, including our results for an infinite capacity queue and an ordinary (non-vacation) queue, by taking the limits.

2. The model

We consider an M/GI/1/N queue, where the input process is Poissonian with rate λ, the service times form a sequence of i.i.d. random variables with distribution function $S(x)$ and N equals the number of waiting places in the queue, including the space for the customer that may be in service. We assume that accepted customers by the system are served by a single server exhaustively, i.e. the server serves the queue continuously until the queue is empty, where a customer is accepted by the system if the number of customers in the system is less than N. Whenever the queue becomes empty the server starts a vacation with distribution function $V(x)$. If the queue is still empty upon his return, he takes another independent vacation with distribution function $V(x)$. We assume furtheron that the service discipline is non-preemptive and the service order is FIFO.

By f_j and h_j we denote the probabilities that j customers arrive during a service time S and a vacation time V, respectively. Hence

$$f_j = \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x} dS(x), \quad j = 0, 1, \cdots \tag{1}$$

$$h_j = \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x} dV(x), \quad j = 0, 1, \cdots. \tag{2}$$

By *idle period* we denote the time between the time instant when the system becomes empty and the time instant when the server starts service again. The probability that j customers arrive (and are accepted) during such an idle period will be denoted by φ_j and is given by

$$\begin{aligned} \varphi_j &= \sum_{l=0}^{\infty} (h_0)^l h_j = \frac{h_j}{1-h_0}, \qquad j = 1, \cdots N-1 \\ \varphi_N &= \sum_{l=0}^{\infty} (h_0)^l h_N^c = \frac{h_N^c}{1-h_0}, \end{aligned} \tag{3}$$

where

$$h_N^c = \sum_{j=N}^{\infty} h_j.$$

Furtheron we define g_k by

$$\begin{aligned} g_k &= f_k, \quad k = 0, 2, 3, \cdots, \\ g_1 &= f_1 - 1 \end{aligned}$$

and c_k by

$$c_0 = \varphi_1 g_0 - 1,$$
$$c_k = \sum_{i=1}^{k+1} \varphi_i g_{k-i+1} + \varphi_k, \quad 1 \le k \le N-2.$$

3. The queue length distribution at a departure epoch

By $\pi_j, j = 0, \cdots, N-1$, we denote the steady state probability that j customers are left in the system at a departure epoch of a customer, i.e. if $t_0, t_1, \cdots$ are the departure epochs of the customers and L_t is the number of customers in the system at time instant t, then

$$\pi_j = \lim_{m\to\infty} \mathbf{P}(L_{t_m} = j), \quad j = 0, \cdots, N-1.$$

It is easy to see, that π_j satisfy the following equations

$$c_k \pi_0 + \sum_{i=1}^{k+1} g_{k+1-i} \pi_i = 0, \quad 0 \le k \le N-2 \tag{4}$$

and the normalization condition

$$\sum_{j=0}^{N-1} \pi_j = 1. \tag{5}$$

Remark 3.1 The above described departure-epoch embedded Markov chain was considered for infinite capacity queues in Cooper [3].

Remark 3.2 In Lee [11], where also vacation completion epochs are considered, the probabilities p_j and q_j were defined as

$$p_j = \lim_{m\to\infty} \mathbf{P}(L_{t_m} = j, \xi_{t_m} = 1), \quad j = 0, \cdots, N-1,$$
$$q_j = \lim_{m\to\infty} \mathbf{P}(L_{t_m} = j, \xi_{t_m} = 0), \quad j = 0, \cdots, N,$$

where $\xi_t = 0$ or 1 corresponding to whether t is a vacation completion epoch ($\xi_t = 0$) or a service completion epoch ($\xi_t = 1$). The probabilities π_j are different from the probabilities p_j. It can be shown that the relationship is given by

$$\pi_j = \frac{p_j}{\sum_{i=0}^{N-1} p_i}, \quad j = 0, \cdots, N-1. \tag{6}$$

As an example consider the case $N = 2$. In the setting of Lee [11] in order to obtain the queue length at a departure epoch the following set of equations has to be solved.

$$p_0 = f_0(p_1 + q_1), \quad p_1 = (1 - f_0)(p_1 + q_1) + q_2$$
$$q_0 = h_0(p_0 + q_0), \quad q_1 = h_1(p_0 + q_0), \quad q_2 = 1 - p_0 - p_1 - q_0 - q_1$$

whereas in our case only the set

$$c_0 \pi_0 + g_0 \pi_1 = 0, \qquad \pi_0 + \pi_1 = 1$$

has to be solved.

We will solve equations (4) and (5) explicitly in the sequel.

Lemma 3.1 *It holds that*

$$\pi_0 = \frac{(-g_0)^{N-1}}{\sum_{j=1}^{N-1}(-g_0)^{N-1-j}c_{N-1-j}a_j + (-g_0)^{N-1}} \tag{7}$$

with

$$\begin{aligned} a_1 &= 1, \\ a_n &= (-g_0)^{n-1} + \sum_{i=1}^{n-1}(-g_0)^{n-1-i}g_{n-i}a_i, \quad 2 \le n \le N-1. \end{aligned} \tag{8}$$

Proof: From equation (4) for $k = N-2$ and equation (5) one can eliminate π_{N-1}. Hence the new set of equations writes as

$$c_k\pi_0 + \sum_{i=1}^{k+1} g_{k+1-i}\pi_i = 0, \quad 0 \le k \le N-3, \tag{9}$$

$$(c_{N-2}a_1 + (-g_0))\pi_0 + \sum_{i=1}^{N-3}(g_{N-1-i}a_1 + (-g_0))\pi_i + a_2\pi_{N-2} = (-g_0). \tag{10}$$

After n eliminations the equations (4) have the following form:

$$c_k\pi_0 + \sum_{i=1}^{k+1} g_{k+1-i}\pi_i = 0, \quad 0 \le k \le N-2-n, \tag{11}$$

$$\left(\sum_{j=1}^{n}(-g_0)^{n-j}c_{N-1-j}a_j + (-g_0)^n\right)\pi_0 + \sum_{i=1}^{N-2-n}\left(\sum_{j=1}^{n}(-g_0)^{n-j}g_{N-i-j}a_j + (-g_0)^n\right)\pi_i + a_{n+1}\pi_{N-1-n} = (-g_0)^n. \tag{12}$$

This can be shown by induction on n, where $n = 1$ is given by (9) and (10). For $n-1$ one has the equations

$$c_k\pi_0 + \sum_{i=1}^{k+1} g_{k+1-i}\pi_i = 0, \qquad 0 \le k \le N-1-n, \tag{13}$$

$$\left(\sum_{j=1}^{n-1}(-g_0)^{n-1-j}c_{N-1-j}a_j + (-g_0)^{n-1}\right)\pi_0 + \sum_{i=1}^{N-1-n}\left(\sum_{j=1}^{n-1}(-g_0)^{n-1-j}g_{N-i-j}a_j + (-g_0)^{n-1}\right)\pi_i + a_n\pi_{N-n} = (-g_0)^{n-1}. \tag{14}$$

Multiplying (14) by $(-g_0)$ and adding to (13) multiplies by a_n for $k = N-1-n$ yields (12).
Hence after $N-1$ manipulations one gets the remaining two equations

$$c_0\pi_0 + g_0\pi_1 = 0,$$

$$\left(\sum_{j=1}^{N-2}(-g_0)^{N-2-j}c_{N-1-j}a_j + (-g_0)^{N-2}\right)\pi_0 + a_{N-1}\pi_1 = (-g_0)^{N-2}$$

from which the assertion follows. □

Lemma 3.2 *The coefficients a_i, defined recursively by (8), are explicitly obtained as*

$$a_1 = 1, \tag{15}$$
$$a_n = \sum_{j=0}^{n-1}(-g_0)^{n-j-1} \sum_{\delta \in A_j^{\leq n-1}} g_{\delta_1} \cdots g_{\delta_j}, \qquad n \geq 2, \tag{16}$$

where $A_j^{\leq k}$ is the set of all j-tuples $\delta = (\delta_1, \cdots, \delta_j)$ with $\delta_i \in \mathbf{N}$ $(i = 1, \cdots, j)$, $\sum_{i=1}^{j} \delta_i \leq k$ for each $k \geq j \geq 1$ and $\sum_{\delta \in A_0^{\leq k}} g_{\delta_0} = 1$ for each $k \geq 0$. By $\mathbf{N}$ we denote the set of natural numbers, i.e. $\mathbf{N} = \{1, 2, \cdots\}$.

Example 3.1 In order to make the definition of $A_j^{\leq k}$ clear, we will give two examples:

$$A_2^{\leq 4} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\},$$
$$A_3^{\leq 4} = \{(1,1,1),(1,1,2),(1,2,1),(2,1,1)\}.$$

Proof of Lemma 3.2: (by induction on n)
For $n = 2$:

$$a_2 = \sum_{j=0}^{2-1}(-g_0)^{2-j-1} \sum_{\delta \in A_j^{\leq 2-1}} g_{\delta_1} \cdots g_{\delta_j} = (-g_0)^1 \sum_{\delta \in A_0^{\leq 1}} g_{\delta_0} + (-g_0)^0 \sum_{\delta \in A_1^{\leq 1}} g_{\delta_1}$$
$$= (-g_0) + g_1 = (-g_0) + g_1 a_1.$$

For a_n one gets

$$\begin{aligned}
a_n &= \sum_{i=1}^{n-1}(-g_0)^{n-1-i} g_{n-i} a_i + (-g_0)^{n-1} \\
&= \sum_{i=1}^{n-1}(-g_0)^{n-1-i} g_{n-i} \sum_{j=0}^{i} \sum_{\delta \in A_j^{\leq i-1}} g_{\delta_1} \cdots g_{\delta_j} (-g_0)^{i-j-1} + (-g_0)^{n-1} \\
&= \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} g_{n-i}(-g_0)^{n-2-j} \sum_{\delta \in A_j^{\leq i-1}} g_{\delta_1} \cdots g_{\delta_j} + (-g_0)^{n-1} \\
&= \sum_{j=0}^{n-2}(-g_0)^{n-2-j} \sum_{i=1}^{n-j-1} g_i \sum_{\delta \in A_j^{\leq n-i-1}} g_{\delta_1} \cdots g_{\delta_j} + (-g_0)^{n-1} \\
&= \sum_{j=0}^{n-2}(-g_0)^{n-2-j} \sum_{\delta \in A_{j+1}^{\leq n-1}} g_{\delta_1} \cdots g_{\delta_{j+1}} + (-g_0)^{n-1} \\
&= \sum_{j=1}^{n-1} \sum_{\delta \in A_j^{\leq n-1}} g_{\delta_1} \cdots g_{\delta_j} (-g_0)^{n-1-j} + (-g_0)^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{\delta \in A_j^{\leq n-1}} g_{\delta_1} \cdots g_{\delta_j} (-g_0)^{n-1-j}.
\end{aligned}$$

□

Lemma 3.3 *It holds that*

$$\pi_0 = \frac{(-g_0)^{N-1}}{\sum_{j=0}^{N-1}(-g_0)^{N-j-1} \sum_{\delta\in B_j^{\leq N-2}} c_{\delta_1} g_{\delta_2}\cdots g_{\delta_j}}, \tag{17}$$

where $B_j^{\leq k}$ *is the set of all* j*-tuples* $\delta = (\delta_1, \cdots, \delta_j)$ *with* $\delta_1 \in \mathbf{N}_0 = \mathbf{N}\cup\{0\}$, $\delta_i \in \mathbf{N}$ $(i = 2, \cdots, j)$, $\sum_{i=1}^{j}\delta_i \leq k$ *for each* $k \geq j-1 \geq 0$ *and* $\sum_{\delta\in B_0^{\leq k}} c_{\delta_0} = 1$ *for each* $k \geq 0$.

Proof: Combining Lemma 3.1 and Lemma 3.2 one gets

$$\begin{aligned}
\pi_0^{-1}(-g_0)^{N-1} &= \sum_{j=1}^{N-1}(-g_0)^{N-j-1}c_{N-1-j}a_j + (-g_0)^{N-1} \\
&= \sum_{j=1}^{N-1}(-g_0)^{N-j-1}c_{N-1-j}\sum_{i=0}^{j-1}\sum_{\delta\in A_i^{\leq j-1}} g_{\delta_1}\cdots g_{\delta_i}(-g_0)^{j-1-i} + (-g_0)^{N-1} \\
&= \sum_{j=1}^{N-1}\sum_{i=0}^{j-1} c_{N-1-j}\sum_{\delta\in A_i^{\leq j-1}} g_{\delta_1}\cdots g_{\delta_i}(-g_0)^{N-2-i} + (-g_0)^{N-1} \\
&= \sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}\sum_{\delta\in A_i^{\leq j-1}} c_{N-1-j}g_{\delta_1}\cdots g_{\delta_i}(-g_0)^{N-2-i} + (-g_0)^{N-1} \\
&= \sum_{i=0}^{N-2}\sum_{j=0}^{N-i-2}\sum_{\delta\in A_i^{\leq N-j-2}} c_j g_{\delta_1}\cdots g_{\delta_i}(-g_0)^{N-2-i} + (-g_0)^{N-1} \\
&= \sum_{i=0}^{N-2}\sum_{\delta\in B_{i+1}^{\leq N-2}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_{i+1}}(-g_0)^{N-2-i} + (-g_0)^{N-1} \\
&= \sum_{i=1}^{N-1}\sum_{\delta\in B_i^{\leq N-2}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_i}(-g_0)^{N-1-i} + (-g_0)^{N-1} \\
&= \sum_{i=0}^{N-1}\sum_{\delta\in B_i^{\leq N-2}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_i}(-g_0)^{N-1-i}.
\end{aligned}$$

□

Lemma 3.4 *For* $1 \leq n \leq N-1$ *it holds that*

$$\pi_n = \frac{\sum_{j=1}^{n}(-g_0)^{n-j}\sum_{\delta\in B_j^{=n-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}}{(-g_0)^n}\pi_0, \tag{18}$$

where $B_j^{=k}$ *is the set of all* j*-tuples* $\delta = (\delta_1, \cdots, \delta_j)$ *with* $\delta_1 \in \mathbf{N}_0$, $\delta_i \in \mathbf{N}$ $(i = 2, \cdots, j)$ *and* $\sum_{i=1}^{j}\delta_i = k$ *for each* $k \geq j-1$.

Proof: (by induction on n)

From (4) for $k=0$ we get

$$c_0\pi_0 + g_0\pi_1 = 0$$

and hence

$$\pi_1 = \frac{c_0}{(-g_0)}\pi_0 = \frac{\sum_{j=1}^{1}(-g_0)^{1-1}\sum_{\delta\in B_j^{=0}} c_{\delta_1}}{(-g_0)^1}\pi_0$$

which proves the assertion for $n=1$. From (4) for $k=n-1$ $(1\le n\le N-1)$ we get

$$\pi_n = \frac{c_{n-1}\pi_0 + \sum_{i=1}^{n-1} g_{n-i}\pi_i}{(-g_0)} \tag{19}$$

and hence

$$\begin{aligned}
(-g_0)^{-n}\pi_n\pi_0^{-1} &= c_{n-1}(-g_0)^{n-1} + \sum_{i=1}^{n-1} g_{n-i}(-g_0)^{n-i-1}\sum_{j=1}^{i}(-g_0)^{i-j}\sum_{\delta\in B_j^{=i-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}\\
&= c_{n-1}(-g_0)^{n-1} + \sum_{i=1}^{n-1} g_{n-i}\sum_{j=1}^{i}(-g_0)^{n-j-1}\sum_{\delta\in B_j^{=i-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}\\
&= c_{n-1}(-g_0)^{n-1} + \sum_{j=1}^{n-1}\sum_{i=j}^{n-1} g_{n-i}(-g_0)^{n-j-1}\sum_{\delta\in B_j^{=i-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}\\
&= c_{n-1}(-g_0)^{n-1} + \sum_{j=2}^{n}\sum_{i=j-1}^{n-1} g_{n-i}(-g_0)^{n-j}\sum_{\delta\in B_{j-1}^{=i-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_{j-1}}\\
&= c_{n-1}(-g_0)^{n-1} + \sum_{j=2}^{n}(-g_0)^{n-j}\sum_{\delta\in B_j^{=n-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}\\
&= \sum_{j=1}^{n}(-g_0)^{n-j}\sum_{\delta\in B_j^{=n-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}
\end{aligned}$$

which completes the proof. □

Combining Lemma 3.3 and Lemma 3.4 yields the following theorem.

Theorem 3.1 *The probabilities that there are n customers left at a departure epoch are explicitly obtained as*

$$\begin{aligned}
\pi_0 &= \frac{(-g_0)^{N-1}}{\sum_{j=0}^{N-1}(-g_0)^{N-j-1}\sum_{\delta\in B_j^{\le N-2}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}},\\
\pi_n &= \frac{\sum_{j=1}^{n}(-g_0)^{n-j}\sum_{\delta\in B_j^{=n-1}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}}{\sum_{j=0}^{N-1}(-g_0)^{n-j}\sum_{\delta\in B_j^{\le N-2}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}}, \quad 1\le n\le N-1.
\end{aligned} \tag{20}$$

Example 3.2 Let $N = 5$. Using (20) one can easily obtain say π_2:

$$\pi_2 = \frac{\sum_{j=1}^{2}(-g_0)^{2-j}\sum_{\delta\in B_j^{=1}} c_{\delta_1}g_{\delta_2}}{\sum_{j=0}^{4}(-g_0)^{2-j}\sum_{\delta\in B_j^{\leq 3}} c_{\delta_1}g_{\delta_2}\cdots g_{\delta_j}}$$

$$= \{-g_0c_1 + c_0g_1\} \times \left\{g_0^2 - g_0(c_0+c_1+c_2+c_3) + c_0(g_1+g_2+g_3) \right.$$
$$\left. + c_1(g_1+g_2) + c_2g_1 - g_0^{-1}\left(c_0(g_1^2+2g_1g_2) + c_1g_1^2\right) + g_0^{-2}c_0g_1^3\right\}^{-1}.$$

Remark 3.3 We can apply the same technique to the finite capacity queue without vacation to obtain the explicit solution for the probabilities that there are n customers left at a departure epoch. This can be easily seen by setting

$$c_0 = g_0 - 1,$$
$$c_k = g_k, \quad k \geq 1.$$

Remark 3.4 Via transform-free method, Niu and Cooper [16] derived the waiting time distribution in terms of σ_k. Here, σ_k is the stationary probability that there are k customers waiting in the queue immediately after a service-start epoch. Note that σ_k is given by

$$\sigma_k = \pi_{k+1} + \pi_0\varphi_{k+1}, \qquad k = 0, \cdots, N-1, \tag{21}$$

and hence given explicitly.

4. The queue length distribution at an arbitrary time in steady state

In this section we will derive the probabilities π_j^*, that there are j customers in the system at an arbitrary time in steady state ($j = 0, \cdots, N$).

Let ρ' be the probability that the server is busy, then the following lemma holds.

Lemma 4.1 *It holds that*

$$\rho' = \lambda(1 - \pi_N^*)\mathbf{E}(S), \tag{22}$$

where π_N^* is the probability that N customers are in the system, and $\mathbf{E}(S)$ is the expected service time.

Proof: We restrict ourselves to only the service facility (excluding the waiting room). By applying the Poisson Arrivals See Time Averages (PASTA) property (see Wolff [19]), we see that π_N^* is also the probability that N customers are in the system just before an arrival epoch. Hence the rate $\lambda(1 - \pi_N^*)$ is the arrival rate of customers accepted by the system, and it is also the throughput of the service facility. The mean number of customers in the service facility is equal to ρ'. Applying Little's law, we then obtain (22). ■

The following theorem will link π_N^* with π_0 which is the probability that no customer is left in the system at a departure epoch.

Theorem 4.1 *The probability that the server is busy (not on vacation) is given by*

$$\rho' = \frac{\mathbf{E}(S)(1-h_0)}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}, \tag{23}$$

where $\mathbf{E}(V)$ *is the expected vacation time.*

Proof: Let us consider two point processes, where the first one is formed by the beginning epochs of busy periods and the second one by the end epochs of busy periods. By λ_{be} and λ_{ee} we denote the rates (expected number of points in a unit time interval) of these two point processes, respectively. Obviously, $\lambda_{be} = \lambda_{ee}$ and it holds that

$$\lambda_{be} = \frac{(1-\rho')(1-h_0)}{\mathbf{E}(V)}, \tag{24}$$

$$\lambda_{ee} = \frac{\rho'\pi_0}{\mathbf{E}(S)}. \tag{25}$$

The first equation can be seen by noting that $(1-\rho')/\mathbf{E}(V)$ is the rate of the point process formed by the time instants where the server returns from a vacation and $1-h_0$ is the probability that the system is not empty anymore, which means that a busy period starts. The second equation can be seen by noting that $\rho'/\mathbf{E}(S)$ is the rate of the point process formed by the departure epochs of customers (i.e. service completion epochs) and π_0 is the probability that no customer is left in the system, which means that a busy period ends. By setting $\lambda_{be} = \lambda_{ee}$ the assertion follows. □

Remark 4.1 There are several ways to prove Theorem 4.1 more rigorously. One way is to consider the relationship

$$\rho' = \frac{\mathbf{E}(B)}{\mathbf{E}(B)+\mathbf{E}(I)}, \tag{26}$$

where B is a typical busy length and I is a typical idle length. Let $S, S_1, S_2, \cdots$ be a sequence of i.i.d. service times with distribution function $S(x)$ and $V, V_1, V_2, \cdots$ be a sequence of i.i.d. vacation times with distribution function $V(x)$. Then,

$$B = \sum_{i=1}^{K_1} S_i \qquad \text{and} \qquad I = \sum_{i=1}^{K_2} V_i,$$

where K_1 (K_2) is the random number of services (vacations) during a busy (idle) period. Clearly, K_1 and K_2 are geometric distributed with parameters π_0 and $1-h_0$, respectively. Applying Wald's identity yields

$$\mathbf{E}(B) = \frac{\mathbf{E}(S)}{\pi_0} \qquad \text{and} \qquad \mathbf{E}(I) = \frac{\mathbf{E}(V)}{1-h_0}. \tag{27}$$

Inserting (27) into (26) completes the proof.

Note also, that (27) can be obtained via λ_{be} and λ_{ee} by the relationship

$$\mathbf{E}(B) = \frac{\rho'}{\lambda_{ee}} \qquad \text{and} \qquad \mathbf{E}(I) = \frac{1-\rho'}{\lambda_{be}}.$$

By using Lemma 4.1 and Theorem 4.1 we can now calculate $(1-\pi_N^*)$, the probability that a customer is being accepted by the system,

$$1-\pi_N^* = \frac{(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}. \tag{28}$$

Because of the PASTA property, we see that π_j^* is also the probability that there are j customers in the system just before an arrival. Thus, the generalized version of Burke's theorem (see Burke [1], Gebhardt [9], Cohen [2], Cooper [4]) is applied to get

$$\pi_j = \frac{\pi_j^*}{1-\pi_N^*}, \quad j = 0, \cdots, N-1. \tag{29}$$

Substituting (28) into (29) we obtain the following theorem.

Theorem 4.2 *The queue length distribution* $\{\pi_j^*; j = 0, \cdots, N\}$ *at an arbitrary time in steady state is obtained as*

$$\pi_j^* = \frac{\pi_j(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}, \quad j = 0, \cdots, N-1 \tag{30}$$

$$\pi_N^* = 1 - \frac{(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}. \tag{31}$$

Remark 4.2 By inserting (20) into (30) and (31) we obtain an explicit solution for the queue length distribution at an arbitrary time in steady state.

Remark 4.3 Equations (30) and (31) are seen to coincide with equation (9) given in Lee [11]. Note that the calculation of $\{\pi_i; i = 0, \cdots, N-1\}$ given by (4) and (5) is simpler than the calculation of $\{p_i; i = 0, \cdots, N-1\}$ in Lee [11], and that the argument for deriving $\{\pi_j^*\}$ is fairly lengthy in Lee [11]. We obtained a more efficient way to calculate the probabilities $\{\pi_i^*; i = 0, \cdots, N\}$.

Remark 4.4 From (30) and (31) we can straightforwardly obtain the Laplace-Stieltjes transform of the waiting time distribution (see Frey and Takahashi [7] for a discussion on it).

5. Concluding remarks

We considered an M/GI/1/N queue with vacation time and exhaustive service discipline. We presented a simple analysis to obtain the queue length distribution *explicitly*. It should be noted that our solution technique can be also used for a finite capacity queue without vacations (see Remark 3.3).

Our results coincide with the results in Lee [11] but require much less computational effort. Furtheron the limits of our quantities (without vacation or infinite waiting room) coincide with known results which can be seen in the following.

1. We will consider the case of an M/GI/1/N queue without vacation. If the vacation period is deterministic, then equations (30) and (31) become

$$\pi_j^*(V) = \frac{\pi_j(V)(1-e^{-\lambda V})\lambda^{-1}}{V\pi_0(V) + \mathbf{E}(S)(1-e^{-\lambda V})}, \quad j = 0, \cdots, N-1 \tag{32}$$

$$\pi_N^*(V) = 1 - \frac{(1-e^{-\lambda V})\lambda^{-1}}{V\pi_0(V) + \mathbf{E}(S)(1-e^{-\lambda V})}, \tag{33}$$

where we write $\pi_j(V)$ ($\pi_j^*(V)$) instead of π_j (π_j^*) to emphasize that the probabilities π_j (π_j^*) depend on V ($j = 0, \cdots, N-1(N)$). By letting $V \to 0$ we obtain

$$\lim_{V\to 0} \pi_j^*(V) = \frac{\pi_j(0)}{\pi_0(0)+\rho}, \quad j = 0, \cdots N-1$$

$$\lim_{V\to 0} \pi_N^*(V) = 1 - \frac{1}{\pi_0(0)+\rho},$$

which coincide with the results given in Gebhardt [9] and Cooper [4].

2. We will consider the case of an M/GI/1/∞ queue with vacation times, i.e. the capacity of the queue is unlimited. As above, we will write $\pi_j(N)$ ($\pi_j^*(N)$) instead of π_j (π_j^*) to emphasize that the probabilities π_j (π_j^*) depend on the capacity N ($j = 0, \cdots, N-1(N)$). By using Burke's theorem (see Burke [1] and Cooper [4]) and the PASTA property (see Wolff [19]) we have to show

$$\lim_{N\to\infty} \pi_j^*(N) = \pi_j(\infty), \quad j = 0, 1, \cdots \tag{34}$$

Following the arguments in the proof of Theorem 4.1 we see that

$$\frac{(1-\rho)(1-h_0)}{\mathbf{E}(V)} = \frac{\rho\pi_0(\infty)}{\mathbf{E}(S)}$$

holds for the infinite capacity queue and hence

$$\frac{\lambda\mathbf{E}(V)\pi_0(\infty)}{1-h_0} = 1-\rho. \tag{35}$$

Hence equation (30) becomes

$$\begin{aligned}\lim_{N\to\infty} \pi_j^*(N) &= \lim_{N\to\infty} \frac{\pi_j(N)(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0(N) + \mathbf{E}(S)(1-h_0)} = \lim_{N\to\infty} \frac{\pi_j(N)}{\dfrac{\lambda\mathbf{E}(V)\pi_0(N)}{1-h_0} + \rho} \\ &= \pi_j(\infty), \quad j = 0, 1, \cdots.\end{aligned}$$

Finally, let us consider a single server queue with finite waiting room and with server vacations under a general vacation policy. The service policy may be arbitrary as long as it is work-conserving and fixed. By Ψ_a we denote the input process (which is a marked point process, where the marks are formed by the service times) of the system, by N_a its arrival process, by $W(t)$ the workload at time t and by $L(t)$ the queue length at time t. We assume that the arrival process is simple and has a finite and positive intensity λ. We assume furtheron that Ψ_a, $\{W(t)\}_{t\in\mathbf{R}}$ and $\{L(T)\}_{t\in\mathbf{R}}$ are jointly stationary under a probability measure $\mathbf{P}$ (see Franken et al. [6] and Miyazawa [15] for details of those definitions). Using the rate conservation law (see Miyazawa [13]) and equations (2.5) and (2.6) of Miyazawa [14] we prove in a further paper that Lemma 4.1 and equation (29) which is given by Gebhardt [9] for the M/G/1/N-queue, are also valid for this more general setting.

For further research, we think about a simple analysis for the M/GI/1/N queue with vacation time and limited service discipline, where our approach will be based on the service completion epochs only.

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