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ON A GENERALIZED M/G/1 QUEUE WITH SERVICE DEGRADATION/ENFORCEMENT

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Abstract A generalized $M/G/1$ queueing system is considered where the efficiency of the server varies as the number of customers served in a busy period increases due to server fatigue or service enforcement. More specifically, the k-th arriving customer within a busy period has the random service requirement V_k where the 0-th customer initiates the busy period and V_k $(k = 0, 1, 2, \cdots)$ are mutually independent but may have different distributions. This model includes an $M/G/1$ queueing model with delayed busy period as a special case where V_k are i.i.d. for $k \geq 1$. Transform results are obtained for the system idle probability at time t, the busy period, and the number of customers at time t given that m customers have left the system at time t since the commencement of the current busy period. The virtual waiting time at time t is also analyzed. A special case that V_k are i.i.d. for $k \geq 2$ is treated in detail, yielding simple and explicit solutions.

1 Introduction

We consider a single server queueing system where customers arrive according to a Poisson process with intensity λ and the service discipline is FIFO. In each busy period, the service time of the k-th arriving customers is denoted by V_k $(k \geq 0)$ where the 0-th customer initiates the busy period. It is assumed that V_k $(k \geq 0)$ are mutually independent but may have different distribution functions $V_k(x)$ $(k \geq 0)$.

There are many queueing situations to which such a model is immediately applicable. For example, consider a case that customers carry i.i.d. service times S with common distribution function $S(x)$ but its service time will be changed as $\alpha_m S$, where α_m is a constant number depending on the number of customers who already left the system since the beginning of the current busy period. When $1 \leq \alpha_m \leq \alpha_{m+1}$ holds for $m \geq 0$, the model may describe a system with gradual server fatigue. On the other hand, when $1 \geq \alpha_m \geq \alpha_{m+1}$, the model may represent a system with service enforcement as the busy period is prolonged. Queueing situations of this sort would naturally arise in analysis of communication protocols in ATM(Asynchronous Transfer Mode) networks, where slots of time-frames would be allocated to voice and data packets dynamically. If we focus on data transmission, the service rate for data packets would change as the corresponding slot allocation varies over a busy period. Specifications of α_m would enable one to analyze a variety of communication protocols.

Another example may be a single server queueing system with i.i.d. service times S where the server takes a vacation whenever j customers are served without interruption, resulting in the expanded service completion time $V + S$ for the $(mj + 1)$-th customer $(m = 1, 2, \cdots)$ within a busy period. The model of this paper also includes an $M/G/1$ queueing system with delayed busy period studied by Welch[2], where V_k are i.i.d. for $k \geq 1$.

The structure of the paper is as follows. In Section 2, the model is formally described and necessary notation is introduced. Transform results are obtained in Section 3 for the system idle probability and the busy period, establishing a relationship between the two. Section 4 analyzes the time-dependent joint distribution of the number of customers in system and the number of customers that have already left the system during the current busy period. The remaining service time and the virtual waiting time are considered in Section 5. A special case that V_k are i.i.d. for $k \geq 2$ is analyzed and simple and concrete results are derived. This is the topic of Section 6.

2 Model Description

Customers arrive at a single server queuing system according to a Poisson process with intensity λ. Let $N(t)$ be the number of customers present in system at time t, including the one in service, if any. Furthermore, let $M(t)$ be the number of customers served completely within the current busy period at time t. If the server is idle at time t, $M(t)$ is defined to be 0. When $M(t) = m$, the service time of a customer who is currently in service is V_m, with distribution function $V_m(x), m = 0, 1, 2, \cdots$. We assume that $V_m(x)$ has the density function $v_m(x)$. Note that a customer whose arrival causes a new busy period is called the 0-th customer of the busy period. We define

$$V_m(x) \equiv \boldsymbol{P}[V_m \leq x], \quad \overline{V}_m(x) \equiv 1 - V_m(x), \tag{2.1}$$

$$v_m(x) \equiv \frac{d}{dx} V_m(x), \tag{2.2}$$

and

$$\eta_m(x) \equiv \frac{v_m(x)}{\overline{V}_m(x)}. \tag{2.3}$$

Suppose that there are no customers in system at $t = 0$, that is $M(0) = 0, N(0) = 0$. The process $\{M(t), N(t)\}$ is not Markov. Let $X(t)$ be the cumulative service given to the customer currently in service if there is a customer in system at time t. If the system is idle at time t, then $X(t) \equiv 0$. $M(t), N(t), X(t)$ are the states just after time t, and hence they are all continuous from the right side. It is obvious that $\{M(t), N(t), X(t)\}$ is a vector valued Markov process. Throughout the paper, we assume that $M(0) = 0, N(0) = 0$ and $X(0) = 0$.

For $t > 0$, let

$$\varepsilon(t) \equiv \boldsymbol{P}[M(t) = 0, N(t) = 0, X(t) = 0], \tag{2.4}$$

and

$$F_{m,n}(x,t) \equiv \boldsymbol{P}[M(t) = m, N(t) = n, X(t) \leq x], \ m = 0, 1, 2, \cdots, \ n = 1, 2, 3, \cdots. \tag{2.5}$$

Thus, $\varepsilon(t)$ denotes the probability that the system is idle at time t, and $\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} F_{m,n}(\infty, t) = 1 - \varepsilon(t)$. We assume that $F_{m,n}(x,t)$ is absolutely continuous with respect to the first variable and write $f_{m,n}(x,t) \equiv \frac{\partial}{\partial x} F_{m,n}(x,t)$.

Considering the event $[M(t + \triangle) = 0, N(t + \triangle) = 0, X(t + \triangle) = 0]$ preceded by the event $[M(t) = 0, N(t) = 0, X(t) = 0]$ or $[M(t) = m, N(t) = 1, X(t) = x]$, $m = 0, 1, 2, \cdots$, where $\triangle$ is positive and sufficiently small, one sees that

$$\varepsilon(t + \triangle) = \{1 - \lambda\triangle\}\varepsilon(t) + \triangle \sum_{m=0}^{\infty} \int_0^t f_{m,1}(x,t)\eta_m(x)dx + o(\triangle), \ t > 0. \tag{2.6}$$

By dividing both sides of Equation (2.6) by $\triangle$ and letting $\triangle \to 0$, together with the initial condition $\varepsilon(0) = 1$, we obtain

$$\frac{d}{dt}\varepsilon(t) = -\lambda\varepsilon(t) + \sum_{m=0}^{\infty}\int_0^t f_{m,1}(x,t)\eta_m(x)dx \ . \tag{2.7}$$

Consider now the situation in which $[M(t) = m, N(t) = n, X(t) = 0]$, i.e., just before t a new service has been started. In this case if $m = 0$ and $n = 1$, then the system is idle at time $t - \triangle$, i.e., $M(t-\triangle) = 0, N(t-\triangle) = 0, X(t-\triangle) = 0$, and a customer arrives during the time interval $[t-\triangle, t]$. For $m = 0, n \geq 2$, $\boldsymbol{P}[M(t) = 0, N(t) = n, X(t) = 0] = 0$. And if $m \geq 1, n \geq 1$ then the system is busy at time $t - \triangle$, and the service of the $(m-1)$-st customer in that busy period has completed during the time interval $[t - \triangle, t]$, i.e., $M(t-\triangle) = M(t) - 1, N(t-\triangle) = N(t) + 1, X(t-\triangle) = x - \triangle$. It follows that

$$f_{0,1}(0,t)\triangle = \lambda\triangle\varepsilon(t-\triangle) + o(\triangle) \ , \tag{2.8}$$

$$f_{0,n}(0,t) = 0 \ , \quad n = 2,3,4,\cdots, \tag{2.9}$$

$$f_{m,n}(0,t)\triangle = \triangle\int_0^t f_{m-1,n+1}(x-\triangle, t-\triangle)\eta_{m-1}(x-\triangle)dx + o(\triangle) \quad m = 1,2,3,\cdots, \ n = 1,2,3,\cdots. \tag{2.10}$$

Proceeding to the limit $\triangle \to 0$, we obtain

$$f_{0,n}(0,t) = \delta_{1,n}\lambda\varepsilon(t), \qquad n = 1,2,\cdots, \quad \text{where } \delta_{1,n} \equiv 1 \text{ for } n = 1, \ \delta_{1,n} \equiv 0 \text{ for } n \neq 1, \tag{2.11}$$

and

$$f_{m,n}(0,t) = \int_0^t f_{m-1,n+1}(x,t)\eta_{m-1}(x)dx, \quad m = 1,2,3,\cdots, \ n = 1,2,3,\cdots. \tag{2.12}$$

Next we consider the situation $[M(t) = m, N(t) = n, X(t) = x]$ and $x > 0$. Since the same customer is in service at time $t - \triangle$, the number of customers who already left the system in the current busy period is not changed during $[t - \triangle, t]$, i.e., $M(t-\triangle) = M(t)$. But for $n = 2, 3, 4, \cdots$ an arrival may have occurred during $[t - \triangle, t]$. And for $n = 1$ the only case $[M(t-\triangle) = M(t), N(t-\triangle) = 1, X(t-\triangle) = x - \triangle]$ arises. Then we have

$$f_{m,1}(x,t)\triangle = \{1-\lambda\triangle\}\{1-\eta_m(x-\triangle)\triangle\}f_{m,1}(x-\triangle, t-\triangle) + 0(\triangle) \ , \quad m = 0,1,2,,\cdots, \tag{2.13}$$

and

$$\begin{aligned} f_{m,n}(x,t)\triangle &= \{1-\lambda\triangle\}\{1-\eta_m(x-\triangle)\triangle\}f_{m,n}(x-\triangle, t-\triangle) \\ &\quad +\lambda\triangle\{1-\eta_m(x-\triangle)\triangle\}f_{m,n-1}(x-\triangle, t-\triangle) + 0(\triangle), \\ &\quad m = 0,1,2,,\cdots, n = 2,3,4,\cdots. \end{aligned} \tag{2.14}$$

By proceeding to the limit $\Delta t \to 0$, we obtain the following partial differential equations :

$$\frac{\partial}{\partial x}f_{m,1}(x,t) + \frac{\partial}{\partial t}f_{m,1}(x,t) = -\{\lambda + \eta_m(x)\}f_{m,1}(x,t), \quad m = 0,1,2,,\cdots, \tag{2.15}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}f_{m,n}(x,t) + \frac{\partial}{\partial t}f_{m,n}(x,t) &= -\{\lambda + \eta_m(x)\}f_{m,n}(x,t) + \lambda f_{m,n-1}(x,t), \\ &\quad m = 0,1,2,,\cdots, n = 2,3,4,\cdots. \end{aligned} \tag{2.16}$$

Equations (2.7), (2.11), (2.12), (2.15) and (2.16) give us all information about the transient behavior of the Markov Process $\{M(t), N(t), X(t)\}$.

By considering the situation $[M(t) = m, N(t) = n, X(t) = x]$, $x > 0$, we see that only arrivals may have occurred and no customers leave the system during the time interval $(t-x, x)$. Hence at time $t-x$ we have the events $[M(t-x) = m, N(t-x) = n-k, X(t-X) = 0]$, $k = 0, 1, 2\cdots, n-1$. Thus we obtain that, by conditioning on the state of this Markov Process at time $t-x$,

$$(2.17) f_{m,n}(x,t) = \sum_{k=0}^{n-1} f_{m,n-k}(0, t-x) e^{-\lambda x} \frac{(\lambda x)^k}{k!} \overline{V}_m(x), m = 0, 1, 2, \cdots, n = 1, 2, 3, \cdots,$$

where for the case of $m = 0$, we recall the equations $f_{0,n}(x,t) = 0$ for $n = 2, 3, 4\cdots$.

Substituting (2.17) into the right hand sides of (2.7) and (2.12) respectively, and using $v_m(x) = \overline{V}_m(x)\eta_m(x)$, we obtain

$$(2.18) \qquad \frac{d}{dt}\varepsilon(t) = -\lambda\varepsilon(t) + \sum_{m=0}^{\infty} \int_0^t f_{m,1}(0, t-x) e^{-\lambda x} v_m(x) dx ,$$

and

$$(2.19) \qquad f_{m,n}(0,t) = \sum_{k=0}^{n} \int_0^t f_{m-1,n-k+1}(0, t-x) e^{-\lambda x} \frac{(\lambda x)^k}{k!} v_{m-1}(x) dx, \quad m = 1, 2, 3, \cdots,\ n = 1, 2, 3, \cdots, x > 0.$$

For notational convenience, we introduce the following functions, transforms, and generating functions.

$$(2.20) \qquad \widehat{v}_m(s) \equiv \int_0^{\infty} e^{-st} v_m(t) dt,\ m = 0, 1, 2, \cdots,$$

$$(2.21) \qquad \widehat{\varepsilon}(s) \equiv \int_0^{\infty} e^{-st} \varepsilon(t) dt,$$

$$(2.22) \qquad \widehat{f}_{m,n}(x,s) \equiv \int_0^{\infty} e^{-st} f_{m,n}(x,t) dt,\ m = 0, 1, 2, \cdots,\ n = 1, 2, 3, \cdots,\ x \geq 0,$$

$$(2.23) \qquad g_{m,n}(t) \equiv \frac{(\lambda t)^n}{n!} v_m(t),\ m = 0, 1, 2, \cdots,\ n = 0, 1, 2, \cdots,$$

$$(2.24) \qquad \widehat{g}_{m,n}(s) \equiv \int_0^{\infty} e^{-st} g_{m,n}(t) dt, m = 0, 1, 2, \cdots,\ n = 0, 1, 2, \cdots,$$

$$(2.25) \qquad \varphi_m(x,t;w) \equiv \sum_{n=1}^{\infty} f_{m,n}(x,t) w^n,\ \ m = 0, 1, 2, \cdots,\ x \geq 0,$$

$$(2.26) \qquad \widehat{\varphi}_m(x,s;w) \equiv \int_0^{\infty} e^{-st} \varphi_m(x,t;w) dt,\ \ m = 0, 1, 2, \cdots,\ x \geq 0.$$

The next proposition plays an important role for studying the transient behavior of the Markov process $[M(t), N(t), X(t)]$ to be discussed in the next section.

Proposition 2.1

$$(2.27) \qquad \widehat{\varepsilon}(s) = \frac{1 + \sum_{m=0}^{\infty} \widehat{f}_{m,1}(0,s) \widehat{v}_m(s+\lambda)}{s+\lambda}$$

Proof: By taking the Laplace transform of both sides of (2.18), one sees that

$$s\hat{\varepsilon}(s) - 1 = -\lambda\hat{\varepsilon}(s) + \sum_{m=0}^{\infty} \hat{f}_{m,1}(0,s)\hat{v}_m(s+\lambda) \tag{2.28}$$

providing the proposition. □

The next proposition also follows in a similar manner from (2.19) and (2.11), and will be used in Section 3.

Proposition 2.2

$$\hat{f}_{m,n}(0,s) = \sum_{k=0}^{n} \hat{f}_{m-1,n-k+1}(0,s)\hat{g}_{m-1,k}(s+\lambda),\ m=1,2,3,\cdots, n=1,2,3,\cdots. \tag{2.29}$$

$$\hat{f}_{0,n}(0,s) = \delta_{1,n}\lambda\hat{\varepsilon}(s),\ n=1,2,3,\cdots. \tag{2.30}$$

Of interest is a recurrence relation for $\hat{\varphi}_m(0,s;w)$ given in the theorem below.

Theorem 2.3

$$\hat{\varphi}_0(0,s;w) = \lambda\hat{\varepsilon}(s)w, \tag{2.31}$$

$$\hat{\varphi}_m(0,s;w) = \frac{1}{w}\hat{\varphi}_{m-1}(0,s;w)\hat{v}_{m-1}(s+\lambda-\lambda w) - \hat{f}_{m-1,1}(0,s)\hat{v}_{m-1}(s+\lambda), \quad m=1,2,3,\cdots. \tag{2.32}$$

Proof: For $x>0$, from (2.17), one has

$$\begin{aligned} \varphi_m(x,t;w) &= \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{n-1} f_{m,n-k}(0,t-x)e^{-\lambda x}\frac{(\lambda x)^x}{k!}\overline{V}_m(x)\right\}w^n \\ &= \sum_{k=0}^{\infty}\left\{\sum_{n=1}^{\infty} f_{m,n}(0,t-x)w^n\right\}e^{-\lambda x}\frac{(\lambda x)^x}{k!}\overline{V}_m(x) \\ &= \varphi_m(0,t-x;w)e^{-\lambda x(1-w)}\overline{V}_m(x),\ m=0,1,2,\cdots,\quad t-x>0. \end{aligned} \tag{2.33}$$

For $x=0, m=0$, from (2.11) , we obtain

$$\varphi_0(0,t;w) = \sum_{n=1}^{\infty} f_{0,n}(0,t)w^n = f_{0,1}(0,t)w = \lambda\varepsilon(t)w. \tag{2.34}$$

For $x=0, m\geq 1$, from (2.12) and (2.33), we see that

$$\begin{aligned} \varphi_m(0,t;w) &= \int_0^t \sum_{n=1}^{\infty} f_{m-1,n+1}(x,t)w^n\eta_m(x)dx \\ &= \frac{1}{w}\int_0^t\left\{\sum_{n=1}^{\infty} f_{m-1,n}(x,t)w^n - f_{m-1,1}(x,t)w\right\}\eta_{m-1}(x)dx \\ &= \frac{1}{w}\int_0^t \varphi_{m-1}(x,t;w)\eta_{m-1}dx - \int_0^t f_{m-1}(x,t)\eta_{m-1}dx, \\ &= \frac{1}{w}\int_0^t\left\{\varphi_{m-1}(0,t-x;w)e^{-\lambda x(1-w)}\overline{V}_{m-1}(x)\right\}\eta_{m-1}(x)dx \\ &\quad - \int_0^t\left\{f_{m-1,1}(0,t-x)e^{-\lambda x}\overline{V}_{m-1}(x)\right\}\eta_{m-1}(x)dx. \end{aligned} \tag{2.35}$$

Taking the Laplace transform of both sides of (2.34) and (2.35) respectively, the theorem follows. □

3 The System Idle Probability and The Busy Period

In this section, we derive the transform results for the system idle probability and the busy period. A preliminary lemma is needed. Let a_k correspond to the number of arrivals during the service time of the k-th customer ($k \geq 0$) in a busy period. Suppose that there is no one waiting in the system when the m-th customer starts receiving the service. (Hence if $a_m = 0$, the current busy period is terminated.) Of interest of a combinatorial nature is a set of $\{a_0, a_1, \cdots, a_m\}$ which realizes the above situation. In order to construct this set, we introduce a set $U_{m,k}$ of sequences of nonnegative integers of length $k+1$ generated recursively on k in the following manner.

[Step 0] $U_{m,0} = \{\{1\}\}$
[Step k] ($k = 1, 2, \cdots, m-1$)

$$\begin{array}{llllll} \text{If} & & \{a^*_{m-k}, & a^*_{m-k+1}, & \cdots, & a^*_{m-1}\} \in U_{m,k-1}, \\ \text{then} & \{a_{m-k-1}, & a_{m-k}, & a_{m-k+1}, & \cdots, & a_{m-1}\} \in U_{m,k}, \end{array}$$

where $a_i = a^*_i$, for $m-k+1 \leq i \leq m-1$, and $a_{m-k-1} + a_{m-k} = a^*_{m-k} + 1$.

The set of original interest is then obtained as $U_{m,m-1}$. For clarity, the example of $m = 3$ is given below.

$$\begin{aligned} U_{3,0} &= \{\{1\}\} \\ U_{3,1} &= \{\{2,0\},\{1,1\}\}, \\ U_{3,2} &= \{\{3,0,0\},\{2,1,0\},\{1,2,0\},\{2,0,1\},\{1,1,1\}\}. \end{aligned}$$

For notational convenience, we decompose the set $U_{m,m-1}$ by the value of a_0. More specifically, $\{a_0, a_1, \cdots a_{m-1}\} \in S_{m,n}$ implies that $a_0 = n$ and $\{a_0, a_1, \cdots a_{m-1}\} \in U_{m,m-1}$. Consequently, one has

$$U_{m,m-1} = \bigcup_{n=1}^{m} S_{m,n} \ , \quad m = 1, 2, 3, \cdots . \tag{3.1}$$

Using these sets $U_{m,m-1}$, the next lemma follows.

Lemma 3.1

$$\widehat{f}_{m,1}(0,s) = \lambda \widehat{\varepsilon}(s) \gamma_m(s+\lambda), \quad m = 0, 1, 2, \cdots, \tag{3.2}$$

where

$$\gamma_m(s) \equiv \begin{cases} 1 & \text{for } m = 0, \\ \sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}} \prod_{k=0}^{m-1} \widehat{g}_{k,a_k}(s) & \text{for } m = 1, 2, 3, \cdots . \end{cases} \tag{3.3}$$

Proof: From (2.30), for $m = 0$ Equation (3.2) holds obviously. For $m \geq 1$, from (2.29), one has

$$\begin{aligned} \widehat{f}_{m,1}(0,s) &= \sum_{k=0}^{1} \widehat{f}_{m-1,2-k}(0,s) \widehat{g}_{m-1,k}(s+\lambda) \\ &= \sum_{\{a_{m-2},a_{m-1}\} \in U_{m,1}} \widehat{f}_{m-1,a_{m-2}}(0,s) \widehat{g}_{m-1,a_{m-1}}(s+\lambda) \\ &= \sum_{\{a_j\}_{j=m-3}^{m-1} \in U_{m,2}} \widehat{f}_{m-2,a_{m-3}}(0,s) \prod_{k=m-2}^{m-1} \widehat{g}_{k,a_k}(s+\lambda) \end{aligned}$$

$$\vdots$$

$$= \sum_{\{a_j\}_{j=0}^{m-1} \in U_{m,m-1}} \widehat{f}_{1,a_0}(0,s) \prod_{k=1}^{m-1} \widehat{g}_{k,a_k}(s+\lambda).$$

To complete the proof we note that $S_{m,n_1} \cap S_{m,n_2} = \emptyset$ when $n_1 \neq n_2$. Replacing $\sum_{\{a_j\}_{j=0}^{m-1} \in U_{m,m-1}}$ in the above equation by $\sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}}$, we obtain

$$\begin{aligned} \widehat{f}_{m,1}(0,s) &= \sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}} \widehat{f}_{1,a_0}(0,s) \prod_{k=1}^{m-1} \widehat{g}_{k,a_k}(s+\lambda) \\ &= \widehat{f}_{0,1}(0,s) \sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}} \widehat{g}_{0,n}(s+\lambda) \prod_{k=1}^{m-1} \widehat{g}_{k,a_k}(s+\lambda) \\ &= \lambda \widehat{\varepsilon}(s) \sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}} \prod_{k=0}^{m-1} \widehat{g}_{k,a_k}(s+\lambda) \\ &= \lambda \widehat{\varepsilon}(s) \gamma_m(s+\lambda) \qquad \text{for } m = 1,2,3,\cdots. \ \square \end{aligned}$$

Substituting (3.2) into the right hand side of (2.27), and solving for $\widehat{\varepsilon}(s)$, we obtain the next theorem.

Theorem 3.2

(3.4)
$$\widehat{\varepsilon}(s) = \frac{1}{s + \lambda - \lambda \sum_{m=0}^{\infty} \gamma_m(s+\lambda) \widehat{v}_m(s+\lambda)}$$

We now turn our attention to the busy period. The busy period analysis can be done along the line of derivation for the ordinary $M/G/1$ system. Let T_{BP} be the busy period, formally defined as

(3.5)
$$T_{BP} \equiv \inf\{t; N(t) = 0 \mid M(0) = 0, N(0) = 1, X(0) = 0\}.$$

We assume that it has the density function denoted by

(3.6)
$$\sigma_{BP}(t) \equiv \boldsymbol{P}[t \leq T_{BP} < t + dt \mid M(0) = 0, N(0) = 1, X(0) = 0],$$

with the Laplace transform

(3.7)
$$\widehat{\sigma}_{BP}(s) \equiv \int_0^{\infty} e^{-st} \sigma_{BP}(t) dt.$$

As for the ordinary $M/G/1$ system, the following relationship between $\widehat{\sigma}_{BP}(s)$ and $\widehat{\varepsilon}(s)$ holds.

Theorem 3.3

(3.8)
$$\widehat{\varepsilon}(s) = \{s + \lambda - \lambda \widehat{\sigma}_{BP}(s)\}^{-1}$$

Proof: We first note that

(3.9)
$$\varepsilon(t+\Delta) = (1 - \lambda\Delta)\varepsilon(t) + \lambda\Delta \int_0^t \sigma(t + \Delta - y)\varepsilon(y) dy + o(\Delta).$$

The first term of the right hand side of this equation describes the case that no arrivals occurred during $[0, \triangle]$. The second term represents the case that an arrival occurs during $[0, \triangle]$ and the busy period initiated by this arrival continues until time $t + \triangle - y$ and no arrivals occur during $[t+\triangle-y, t+\triangle]$. Letting $\triangle \to 0$, Equation (3.9) leads to the following differential equation :

$$\frac{d}{dt}\varepsilon(t) = -\lambda\varepsilon(t) + \lambda\int_0^t \sigma_{BP}(t-y)\varepsilon(y)dy. \tag{3.10}$$

By taking the Laplace transform with $\varepsilon(0) = 1$, we obtain

$$-1 + s\widehat{\varepsilon}(s) = -\lambda\widehat{\varepsilon}(s) + \lambda\widehat{\sigma}_{BP}(s)\widehat{\varepsilon}(s). \tag{3.11}$$

Solving for $\widehat{\varepsilon}(s)$ completes the proof. □

From Theorem 3.2 and Theorem 3.3, we obtain the next corollary immediately.

Corollary 3.4

$$\widehat{\sigma}_{BP}(s) = \sum_{m=0}^{\infty}\gamma_m(s+\lambda)\widehat{v}_m(s+\lambda) \tag{3.12}$$

In (3.12), the right hand side is formed by conditioning on m, the number of customers who arrive during a busy period, i.e., including a customer who arrives to an empty queue and starts this busy period, the busy period ends after having $m+1$ customers served. Hence $\gamma_m(s+\lambda)\widehat{v}_m(s+\lambda)$ is the Laplace transform of the busy period distribution conditioned on m. To interpret (3.12), we recall the definition of $\gamma_m(s)$ in (3.3). From (2.23) and (2.24), one has

$$\gamma_0(s+\lambda) = 1, \tag{3.13}$$

and

$$\begin{aligned}\gamma_m(s+\lambda) &= \sum_{n=1}^{m}\sum_{\{a_i\}_{i=0}^{m-1}\in S_{m,n}}\prod_{k=0}^{m-1}\widehat{g}_{k,a_k}(s+\lambda)\\ &= \sum_{n=1}^{m}\sum_{\{a_i\}_{i=0}^{m-1}\in S_{m,n}}\prod_{k=0}^{m-1}\int_0^\infty e^{-(s+\lambda)t}\frac{(\lambda t)^{a_k}}{a_k!}v_k(t)dt, \quad m = 1,2,3,\cdots.\end{aligned} \tag{3.14}$$

Suppose that $a_0 = n$ customers arrive during the service time of the 0-th customer in this busy period. For $\{a_0 = n, a_1, \cdots, a_{m-1}\} \in S_{m,n}$, a_k customers arrive during the service time of the k-th customer $(1 \le k \le m-1)$ with probability $\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^{a_k}}{a_k!}v_k(t)dt$. The probalistic meaning of (3.14) is then clear.

We are now in a position to find the limit of the system idle probability $\varepsilon(t)$ as $t \to \infty$, and the mean busy period $\boldsymbol{E}[T_{BP}] \equiv \int_0^\infty t\sigma_{BP}(t)dt$.

Theorem 3.5

$$\lim_{t\to\infty}\varepsilon(t) = \frac{1}{1+\lambda\boldsymbol{E}[T_{BP}]} \tag{3.15}$$

$$\begin{aligned}\boldsymbol{E}[T_{BP}] &= \boldsymbol{E}[V_0]\\ &+ \sum_{m=1}^{\infty}\lambda^m\sum_{n=1}^{m}\sum_{\substack{\{a_i\}_{i=0}^{m-1}\in S_{m,n}\\ a_m=0}}\sum_{j=0}^{m}\prod_{\substack{k=0\\k\neq j}}^{m}\left(\int_0^\infty e^{-\lambda t}\frac{t^{a_k}v_k(t)}{a_k!}dt\right)\left(\int_0^\infty e^{-\lambda t}\frac{t^{a_j+1}v_j(t)}{a_j!}dt\right)\end{aligned} \tag{3.16}$$

Proof: Using Theorem 3.3 and the L'Hôpital's rule, one has

$$\begin{aligned}\lim_{t\to\infty}\varepsilon(t) &= \lim_{s\downarrow 0} s\widehat{\varepsilon}(s) = \lim_{s\downarrow 0}\frac{s}{s+\lambda-\lambda\sigma_{BP}(s)} \\ &= \lim_{s\downarrow 0}\frac{1}{1-\lambda(\frac{\partial}{\partial s}\sigma_{BP}(s))} = \frac{1}{1+\lambda\boldsymbol{E}[T_{BP}]}.\end{aligned}$$

The second statement is immediate from Corollary 3.4 and the relation $\boldsymbol{E}[T_{BP}] = -\lim_{s\downarrow 0}\frac{\partial}{\partial s}\sigma_{BP}(s)$.□

4 Time-Dependent Joint Distribution $\boldsymbol{P}[M(t)=m, N(t)=n]$

In this section we analyze the time-dependent joint distribution

$$\begin{aligned}p_{m,n}(t) &= \boldsymbol{P}[M(t)=m, N(t)=n] \\ &= \int_0^\infty f_{m,n}(x,t)dx.\end{aligned} \tag{4.1}$$

The corresponding generating functions and their Laplace transforms are denoted by

$$\begin{aligned}\pi_0(t;w) &\equiv \sum_{n=0}^{\infty} p_{0,n}(t)w^n \\ &= p_{0,0}(t) + \int_0^\infty \sum_{n=1}^{\infty} f_{0,n}(x,t)w^n dx \\ &= \varepsilon(t) + \int_0^\infty \varphi_0(x,t;w)dx,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\pi_m(t;w) &\equiv \sum_{n=1}^{\infty} p_{m,n}(t)w^n \\ &= \int_0^\infty \varphi_m(x,t;w)dx, \qquad m=1,2,3,\cdots,\end{aligned} \tag{4.3}$$

and

$$\widehat{\pi}_m(s;w) \equiv \int_0^\infty e^{-st}\pi_m(t;w)dt, \qquad m=0,1,2,\cdots. \tag{4.4}$$

We have already determined $\widehat{\varepsilon}(s)$ in Theorem 3.2. In the next theorem, $\widehat{\pi}_m(s;w)$ is obtained using $\widehat{\varepsilon}(s)$.

Theorem 4.1

$$\widehat{\pi}_0(s;w) = \widehat{\varepsilon}(s) + \lambda\widehat{\varepsilon}(s)w\frac{1-\widehat{v}_0(s+\lambda-\lambda w)}{s+\lambda-\lambda w} \tag{4.5}$$

$$\begin{aligned}\widehat{\pi}_m(s;w) &= \lambda\widehat{\varepsilon}(s)\left\{w\prod_{k=0}^{m-1}\frac{\widehat{v}_k(s+\lambda-\lambda w)}{w}\right. \\ &\qquad \left. -\sum_{k=0}^{m-1}\gamma_k(s+\lambda)\widehat{v}_k(s+\lambda)\prod_{i=k+1}^{m-1}\frac{\widehat{v}_i(s+\lambda-\lambda w)}{w}\right\} \\ &\qquad \times\frac{1-\widehat{v}_m(s+\lambda-\lambda w)}{s+\lambda-\lambda w}, \quad \text{for } m=1,2,3,\cdots\end{aligned} \tag{4.6}$$

Proof: For $m \geq 1$, from (2.33), it can be seen that

$$\begin{aligned}\int_0^\infty e^{-st}\pi_m(t;w)dt &= \int_0^\infty e^{-st}\left(\int_0^\infty \varphi_m(x,t;w)dx\right)dt \\ &= \int_0^\infty e^{-st}\int_0^\infty \varphi_m(0,t-x;w)e^{-\lambda x(1-w)}\overline{V}_m(x)dxdt \\ &= \int_0^\infty e^{-st}\varphi_m(0,t;w)dt\int_0^\infty e^{-(s+\lambda-\lambda w)x}\overline{V}_m(x)dx\end{aligned}$$

Similarly for $m = 0$, we obtain

$$\widehat{\pi}_0(s;w) = \widehat{\varepsilon}(s) + \widehat{\varphi}_0(0,s;w)\frac{1-\widehat{v}_o(s+\lambda-\lambda w)}{s+\lambda-\lambda w}, \tag{4.7}$$

and

$$\widehat{\pi}_m(s;w) = \widehat{\varphi}_m(0,s;w)\frac{1-\widehat{v}_m(s+\lambda-\lambda w)}{s+\lambda-\lambda w}, \quad m = 1,2,3,\cdots. \tag{4.8}$$

Using the recurrence relation (2.32) and (3.2), for $m \geq 1$, one has

$$\begin{aligned}\widehat{\varphi}_m(0,s;w) &= \widehat{\varphi}_{m-1}(0,s;w)\frac{\widehat{v}_{m-1}(s+\lambda-\lambda w)}{w} - \lambda\widehat{\varepsilon}(s)\gamma_{m-1}(s+\lambda)\widehat{v}_{m-1}(s+\lambda) \\ &= \left\{\widehat{\varphi}_{m-2}(0,s;w)\frac{\widehat{v}_{m-2}(s+\lambda-\lambda w)}{w} - \lambda\widehat{\varepsilon}(s)\gamma_{m-2}(s+\lambda)\widehat{v}_{m-2}(s+\lambda)\right\}\frac{\widehat{v}_{m-1}(s+\lambda-\lambda w)}{w} \\ &\quad -\lambda\widehat{\varepsilon}(s)\gamma_{m-1}(s+\lambda)\widehat{v}_{m-1}(s+\lambda) \\ &\quad \vdots \\ &= \widehat{\varphi}_0(0,s;w)\prod_{k=0}^{m-1}\frac{\widehat{v}_k(s+\lambda-\lambda w)}{w} - \sum_{k=0}^{m-1}\lambda\widehat{\varepsilon}(s)\gamma_k(s+\lambda)\widehat{v}_k(s+\lambda)\prod_{i=k+1}^{m-1}\frac{\widehat{v}_i(s+\lambda-\lambda w)}{w}.\end{aligned}$$

From (2.34), for $m \geq 0$, it follows that

$$\begin{aligned}\widehat{\varphi}_m(0,s;w) &= \lambda\widehat{\varepsilon}(s)\left\{w\prod_{k=0}^{m-1}\frac{\widehat{v}_k(s+\lambda-\lambda w)}{w}\right. \\ &\quad \left.-\sum_{k=0}^{m-1}\gamma_k(s+\lambda)\widehat{v}_k(s+\lambda)\prod_{i=k+1}^{m-1}\frac{\widehat{v}_i(s+\lambda-\lambda w)}{w}\right\}.\end{aligned} \tag{4.9}$$

where the empty sum $\sum_{k=0}^{-1}$ is defined as 0 and the empty product $\prod_{k=0}^{-1}$ is defined as 1. Substituting (4.9) into (4.7) and (4.8), the proof of this theorem is complete. □

5 Remaining Service Time and Virtual Waiting Time

Now we analyze the time-dependent behavior of the joint process of $M(t)$, $N(t)$ and the remaining service time $X_+(t)$. For $m \geq 0$ we define

$$H_{m,n}(v,t) \equiv \boldsymbol{P}[M(t)=m, N(t)=n, X_+(t) \leq v], \tag{5.1}$$

$$h_{m,n}(v,t) \equiv \frac{\partial}{\partial v}H_{m,n}(v,t), \tag{5.2}$$

and

$$\widehat{\Pi}^*_m(u,s;w) \equiv \int_0^\infty dt\; e^{-st}\int_0^\infty e^{-uv}\sum_{n=1}^\infty w^n h_{m,n}(v,t)dv. \tag{5.3}$$

Theorem 5.1

$$(5.4)\quad \widehat{\Pi}^*_m(u,s;w) = \frac{\lambda\widehat{\varepsilon}(s)\{\widehat{v}_m(u)-\widehat{v}_m(s+\lambda-\lambda w)\}}{s-u+\lambda-\lambda w}\times\left\{w\prod_{k=0}^{m-1}\frac{\widehat{v}_k(s+\lambda-\lambda w)}{w} - \sum_{k=0}^{m-1}\gamma_k(s+\lambda)\widehat{v}_k(s+\lambda)\prod_{i=k+1}^{m-1}\frac{\widehat{v}_i(s+\lambda-\lambda w)}{w}\right\}$$

for $m=0,1,2,\cdots$

Proof: Substituting $h_{m,n}(v,t)=\int_0^t f_{m,n}(y,t)\frac{v_m(y+v)}{\overline{V}_m(y)}dy$ into (5.3), and using (2.25) and (2.33), one has

$$\begin{aligned}(5.5)\quad \widehat{\Pi}^*_m(u,s;w) &= \int_0^\infty dt\, e^{-st}\int_0^t dy\sum_{n=1}^\infty w^n\frac{f_{m,n}(y,t)}{\overline{V}_m(y)}\int_0^\infty e^{-uv}v_m(y+v)dv\\ &= \int_0^\infty dt\, e^{-st}\int_0^t dy\,\frac{\varphi_m(y,t;w)}{\overline{V}_m(y)}\int_0^\infty e^{-uv}v_m(y+v)dv\\ &= \int_0^\infty dt\, e^{-st}\int_0^t dy\,\varphi_m(0,t-y;w)e^{-\lambda(1-w)y}\int_y^\infty e^{-u(z-y)}v_m(z)dz\\ &= \widehat{\varphi}_m(0,s;w)\int_0^\infty dy\, e^{-(s-u+\lambda-\lambda w)y}\int_y^\infty e^{-uz}v_m(z)dz\\ &= \widehat{\varphi}_m(0,s;w)\frac{\widehat{v}_m(u)-\widehat{v}_m(s+\lambda-\lambda w)}{s-u+\lambda-\lambda w}.\end{aligned}$$

Substituting (4.9) into the above equation, the theorem follows. □

Next we consider the virtual waiting time or the unfinished work $W(t)$ at time t. The joint distribution function $W_m(x,t)$ of $W(t)$ and $M(t)$ for $m\geq 0$, and its LST are defined by

$$(5.6)\quad W_m(x,t) \equiv \boldsymbol{P}[W(t)\leq x, M(t)=m],\ x\geq 0,$$

$$(5.7)\quad W^*_m(u,t) \equiv \int_0^\infty e^{-ux}d_xW_m(x,t)$$

From the initial condition $N(0)=0$, we have $W(0)=0$. Moreover, we note that

$$(5.8)\quad W_m(0,t)=\begin{cases}\varepsilon(t)=\boldsymbol{P}[N(t)=0], & \text{for } m=0,\\ 0, & \text{for } m\geq 1.\end{cases}$$

We recall that $\overline{V}_m(y)=\int_y^\infty v_m(z)dz$ and $\delta_{m,0}=1$ if $m=0$ and $\delta_{m,0}=0$ otherwise.

Theorem 5.2

$$(5.9)\quad W^*_m(u,t) = \varepsilon(t)\delta_{m,0}+\sum_{n=1}^\infty\prod_{k=m+1}^{m+n-1}\widehat{v}_k(u)\int_0^t dy\, e^{uy}\frac{f_{m,n}(y,t)}{\overline{V}_m(y)}\int_y^\infty e^{-uv}v_m(v)dv,$$

$m=0,1,\cdots,$

Proof: Conditioning on $X(t)=y$, we see that

$$W_m(x,t)=\varepsilon(t)\delta_{m,0}+\sum_{n=1}^\infty\int_0^t \boldsymbol{P}\left[V_m-y+\sum_{k=m+1}^{m+n-1}V_k\leq x\mid V_m\geq y\right]f_{m,n}(y,t)dy.$$

It then follows that

(5.10) $W_m(x,t)$

$$= \varepsilon(t)\delta_{m,0} + \sum_{n=1}^{\infty}\int_0^t dy\ \frac{f_{m,n}(y,t)}{\overline{V}_m(y)}\int_y^{\infty} \boldsymbol{P}\left[\sum_{k=m+1}^{m+n-1} V_k \le x+y-v\right] v_m(v)dv.$$

By taking the Laplace transform of (5.10), with mutual independence of V_k, the theorem follows. □

6 Special Cases

When $V_k(k \ge 0)$ are i.i.d., our model reduces to the ordinary $M/G/1$ system, which we call Case A. If $V_k(k \ge 1)$ are i.i.d., then the model coincides with the delayed busy period for $M/G/1$, originally studied by Welch[2]. This case is called Case B. In this section, we extend Case B by assuming that $V_k(k \ge 2)$ are i.i.d., which we call Case C. The busy period and its transform for Case X are denoted by T_{BP_X} and $\hat{\sigma}_{BP_X}(s)$ respectively for $X = A, B, C$. The primary purpose of this section is to express $\hat{\sigma}_{BP_C}(s)$ in terms of $\hat{\sigma}_{BP_A}(s)$ and $\hat{\sigma}_{BP_B}(s)$.

For notational convenience, we denote the concatenation of two sequences of integers $\boldsymbol{a} = \{a_0, a_1, \cdots, a_i\}$ and $\boldsymbol{b} = \{b_0, b_1, \cdots, b_j\}$ by

$$\boldsymbol{a}(+)\boldsymbol{b} \equiv \{a_0, a_1, \cdots, a_i, b_0, b_1, \cdots, b_j\}.$$

We also denote the truncation of $\boldsymbol{a}$ with the first term dropped by $\boldsymbol{a}^*$, i.e.,

$$\boldsymbol{a}^* \equiv \{a_1, \cdots, a_i\}.$$

The notation is extended in a natural way to similar set operations where

$$A(+)B \equiv \{\boldsymbol{a}(+)\boldsymbol{b} \mid \boldsymbol{a} \in A, \boldsymbol{b} \in B\},$$

and

$$A^* \equiv \{\boldsymbol{a}^* \mid \boldsymbol{a} \in A\}.$$

For the definition of $S_{m,n}$ given in (3.1), we see that $S_{m,n}(+)\{0\}$ is the set of sequences (of length $m+1$) of customer arrivals in individual service times when the busy period ends after having $m+1$ customers served and n customers arrive during the service time of the 0-th customer. Accordingly,

(6.1) $$S_n = \bigcup_{m=n}^{\infty} S_{m,n}(+)\{0\}$$

is the set of such sequences for all busy periods having n customer arrivals during the service time of the 0-th customer. We note that $S_{m,0} = \emptyset$ so that

(6.2) $$S_0 = \{0\}$$

With this interpretation, it is clear that S_n can be decomposed as

(6.3) $$S_n = \{n\}(+)\bigcup_{j=0}^{\infty} S_j(+)S_{n-1}^*, \quad n = 1, 2, \cdots.$$

We are now in a position to prove the following theorem.

Theorem 6.1

$$\widehat{\sigma}_{BP_C}(s)=\widehat{v}_0(s+\lambda)+\frac{\widehat{\sigma}_{BP_B}(s)}{\widehat{\sigma}_{BP_A}(s)}\{\widehat{v}_0(s+\lambda-\lambda\widehat{\sigma}_{BP_A}(s))-\widehat{v}_0(s+\lambda)\} \tag{6.4}$$

Proof: Let $[\boldsymbol{a}]$ denote the largest subscript of the sequence $\boldsymbol{a}$ i.e., when $\boldsymbol{a}=\{a_0,a_1,\cdots,a_m\}$, $[\boldsymbol{a}]=m$. Changing the order of the summations in (3.14) and using (6.3), we can write

$$\begin{aligned}
\widehat{\sigma}_{BP_C}(s) &= \sum_{m=0}^{\infty}\gamma_m(s+\lambda)\widehat{v}_m(s+\lambda)=\widehat{g}_{0,0}(s+\lambda)+\sum_{n=1}^{\infty}\sum_{\boldsymbol{a}\in S_n}\prod_{k=0}^{m}\widehat{g}_{k,a_k}(s+\lambda)\\
&= \widehat{g}_{0,0}(s+\lambda)+\sum_{n=1}^{\infty}\widehat{g}_{0,n}(s+\lambda)\left(\widehat{g}_{1,0}(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{k,b_k}(s+\lambda)\right)\\
&\quad\times\left(\sum_{\boldsymbol{c}\in S^*_{n-1}}\prod_{l=1}^{[\boldsymbol{c}]}\widehat{g}_{c_l}(s+\lambda)\right)\\
&= \widehat{g}_{0,0}(s+\lambda)+\sum_{n=1}^{\infty}\widehat{g}_{0,n}(s+\lambda)\left(\widehat{g}_{1,0}(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{k,b_k}(s+\lambda)\right)\\
&\quad\times\left(\widehat{g}_0(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{b_k}(s+\lambda)\right)\left(\sum_{\boldsymbol{c}\in S^*_{n-2}}\prod_{l=1}^{[\boldsymbol{c}]}\widehat{g}_{c_l}(s+\lambda)\right).
\end{aligned} \tag{6.5}$$

Since $\widehat{g}_{k,a_k}(s+\lambda)=\widehat{g}_{a_k}(s+\lambda)$ for $k\geq 2$, we obtain

$$\begin{aligned}
\widehat{\sigma}_{BP_C}(s) &= \widehat{g}_{0,0}(s+\lambda)+\sum_{n=1}^{\infty}\widehat{g}_{0,n}(s+\lambda)\left(\widehat{g}_{1,0}(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{k,b_k}(s+\lambda)\right)\\
&\quad\times\left(\widehat{g}_0(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{b_k}(s+\lambda)\right)^{n-1}
\end{aligned} \tag{6.6}$$

Setting $\widehat{g}_{i,n}(s+\lambda)\Rightarrow\widehat{g}_{i+1,n}(s+\lambda)$ in (6.5), we notice that

$$\widehat{g}_{1,0}(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{k,b_k}(s+\lambda)=\widehat{\sigma}_{BP_B}(s). \tag{6.7}$$

In the same manner,

$$\widehat{g}_0(s+\lambda)+\sum_{j=1}^{\infty}\sum_{\boldsymbol{b}\in S_j}\prod_{k=1}^{[\boldsymbol{b}]}\widehat{g}_{b_k}(s+\lambda)=\widehat{\sigma}_{BP_A}(s). \tag{6.8}$$

Substituting (6.7) and (6.8) into (6.6), we then obtain

$$\begin{aligned}
\widehat{\sigma}_{BP_C}(s) &= \widehat{g}_{0,0}(s+\lambda)+\sum_{n=1}^{\infty}\widehat{g}_{0,n}(s+\lambda)\widehat{\sigma}_{BP_B}(s)\left(\widehat{\sigma}_{BP_A}(s)\right)^{n-1}\\
&= \int_0^{\infty}e^{-(s+\lambda)t}v_0(t)dt+\frac{\widehat{\sigma}_{BP_B}(s)}{\widehat{\sigma}_{BP_A}(s)}\int_0^{\infty}e^{-(s+\lambda)t}v_0(t)\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\widehat{\sigma}_{BP_A}(s)^n dt\\
&= \widehat{v}_0(s+\lambda)+\frac{\widehat{\sigma}_{BP_B}(s)}{\widehat{\sigma}_{BP_A}(s)}\{\widehat{v}_0(s+\lambda-\lambda\widehat{\sigma}_{BP_A}(s))-\widehat{v}_0(s+\lambda)\} \qquad \square
\end{aligned}$$

The next corollary is immediate as special cases of Theorem 6.1.

Corollary 6.2

(6.9) $$\hat{\sigma}_{BP_A}(s) = \hat{v}(s+\lambda-\lambda\hat{\sigma}_{BP_A}(s))$$

(6.10) $$\hat{\sigma}_{BP_B}(s) = \hat{v}_0(s+\lambda-\lambda\hat{\sigma}_{BP_A}(s))$$

For the mean busy period in each case, we have the next corollary.

Corollary 6.3

(6.11) $$E[T_{BP_A}] = \frac{E[V]}{1-\lambda E[V]} \quad \text{if } \lambda E[V]<1$$

(6.12) $$E[T_{BP_B}] = \frac{E[V_0]}{1-\lambda E[V]} \quad \text{if } E[V_0]<\infty,\ \lambda E[V]<1,$$

(6.13) $$E[T_{BP_C}] = \frac{E[V_0]+(1-\hat{v}_0(\lambda))(E[V_1]-E[V])}{1-\lambda E[V]}$$
$$\text{if } E[V_0]<\infty,\ E[V_1]<\infty,\ \lambda E[V]<1$$

Proof: Finding the derivative of both sides of (6.4) with respect to s, and proceeding to the limit $s\downarrow 0$ yield,

(6.14) $$E[T_{BP_C}] = (E[T_{BP_A}]-E[T_{BP_B}])(\hat{v}_0(\lambda)-1)+E[V_0](1+\lambda E[T_{BP_A}])$$

For $E[T_{BP_A}]$ and $E[T_{BP_B}]$, as special cases of (6.14), we have

(6.15) $$E[T_{BP_A}] = E[V]+\lambda E[V]E[T_{BP_A}]$$

(6.16) $$E[T_{BP_B}] = E[V_0]+\lambda E[V_0]E[T_{BP_A}]$$

These yield the first two assertions in this corollary. Substituting (6.11), (6.12) into (6.14) yields (6.13). □

Now we observe the limiting behavior of the system idle probability and the generating function of the distribution of the queue size in Case C as time $t\to\infty$.

Theorem 6.4

(6.17) $$\lim_{t\to\infty}\varepsilon(t) = \frac{1-\lambda E[V]}{1+\lambda\{E[V_0]-E[V]+(1-\hat{v}_0(\lambda))(E[V_1]-E[V])\}},$$

(6.18) $$\lim_{t\to\infty}\sum_{n=0}^{\infty}\boldsymbol{P}[N(t)=n \mid N(0)=0]w^n$$
$$= \frac{1-\lambda E[V]}{1+\lambda\{E[V_0]-E[V]+(1-\hat{v}_0(\lambda))(E[V_1]-E[V])\}}$$
$$\times\left\{\frac{(w\hat{v}_0(\lambda-\lambda w)-\hat{v}(\lambda-\lambda w))}{(w-\hat{v}(\lambda-\lambda w))}\right.$$
$$\left.+\frac{(\hat{v}_1(\lambda-\lambda w)-\hat{v}(\lambda-\lambda w))(\hat{v}_0(\lambda-\lambda w)-\hat{v}_0(\lambda)+w\hat{v}_0(\lambda))}{(1-w)(w-\hat{v}(\lambda-\lambda w))}\right\}.$$

Proof: From Theorem 3.5 and (6.13), one has

(6.19) $$\lim_{t\to\infty}\varepsilon(t) = \lim_{s\downarrow 0}s\hat{\varepsilon}(s)$$
$$= \frac{1}{1+\lambda E[T_{BP_C}]}$$
$$= \frac{1-\lambda E[V]}{1+\lambda\{E[V_0]-E[V]+(1-\hat{v}_0(\lambda))(E[V_1]-E[V])\}}.$$

From Theorem 4.1, after some algebra, it follows that

$$(6.20)\quad \lim_{t\to\infty}\sum_{n=0}^{\infty}\boldsymbol{P}[N(t)=n\mid N(0)=0]w^n=\lim_{t\to\infty}\sum_{m=0}^{\infty}\pi_m(t;w)=\lim_{s\downarrow 0}s\sum_{m=0}^{\infty}\widehat{\pi}_m(s;w)$$

$$=\left(\lim_{s\downarrow 0}s\widehat{\varepsilon}(s)\right)\left\{\frac{(w\widehat{v}_0(\lambda-\lambda w)-\widehat{v}(\lambda-\lambda w))}{(w-\widehat{v}(\lambda-\lambda w))}+\frac{(\widehat{v}_1(\lambda-\lambda w)-\widehat{v}(\lambda-\lambda w))(\widehat{v}_0(\lambda-\lambda w)-\widehat{v}_0(\lambda)+w\widehat{v}_0(\lambda))}{(1-w)(w-\widehat{v}(\lambda-\lambda w))}\right\}.$$

Substituting (6.19) into (6.20), we obtain the generating function of the distribution of queue size as $t\to\infty$. □

Remark 6.5 *We note that we obtain the same results as Welch's model(Case B) in [2] by setting* $\widehat{v}_1(s)=\widehat{v}(s)$ *and* $V_1=V$ *in theorem 6.4. Moreover, if we set* $\widehat{v}_0(s)=\widehat{v}_1(s)=\widehat{v}(s)$ *and* $V_0=V_1=V$ *in theorem 6.4, the results of the ordinary* $M/G/1$ *(Case A) are immediately derived.*

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