

## PROPERTIES OF A POSITIVE RECIPROCAL MATRIX AND THEIR APPLICATION TO AHP

Shunsuke Shiraishi  
*Toyama University*

Tsuneshi Obata  
*Oita University*

Motomasa Daigo  
*Doshisha University*

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*Abstract* The characteristic polynomial of a positive reciprocal matrix has some noteworthy properties. They are deeply related to the notion of consistency of a pairwise comparison matrix of AHP. Based on the results, we propose a method for estimating a missing entry of an incomplete pairwise comparison matrix.

### 1. Introduction

In AHP (Analytic Hierarchy Process), positive reciprocal matrices appear as pairwise comparison matrices [6, 7, 8, 10]. It is a main ingredient which supports analytical feature of AHP. Based on the pairwise comparison matrix obtained by oral questioning, AHP estimates a priority vector of factors involved. By Saaty's eigenvector method, one computes the priority vector. However, it is a hard task to solve the eigensystem exactly, because it requires solving an algebraic equation of high degree. Since the eigenvector method requires only the principal eigenvector (Perron-Frobenius vector) of the matrix, one practically uses the power method in general [10]. By the recent development of high-performance computer environment, any decision maker can use the eigenvector method easily. On the other hand, attentions to the characteristic polynomial of the pairwise comparison matrix seem to have been paid little. It seems that the existence of the power method has been limiting the investigation of the characteristic polynomial. If one examines the characteristic polynomial in detail, one sees that it is deeply related to the notion of consistency of the pairwise comparison matrix. In this paper, we shed a light on this relationship.

The eigenvector method also yields a measure for inconsistency. The degree of inconsistency is measured by the principal eigenvalue  $\lambda_{\max}$ . If  $A$  is a pairwise comparison matrix of the size  $n$ , it is known that  $\lambda_{\max} \geq n$  and  $A$  is consistent if and only if  $\lambda_{\max} = n$  [10]. Hence, one sees the consistency by the quantity  $\lambda_{\max} - n$ . Normalizing by the size of the matrix, the consistency index (C.I.) is defined by

$$\text{C.I.} = \frac{\lambda_{\max} - n}{n - 1}.$$

Decision makers are required to do a pairwise comparison so that C.I. would not be as far from 0 as possible. In Section 2, we show that it is completely determined by the coefficient of the degree  $n - 3$  of the characteristic polynomial whether C.I. = 0 or not. Based on this fact, in Section 3, we propose a new method for estimating a missing datum of an incomplete pairwise comparison matrix.

When the pairwise comparison matrix has a missing entry, it is an important subject to estimate a priority vector from the incomplete matrix. Several methods have been proposed

for this subject (see [2, 9] and references therein). It seems that, in principle, any of such method aims to make C.I. good. We would also like to aim to minimize C.I. However, it is not an easy matter to minimize C.I. exactly. Instead of minimizing C.I., we propose a heuristic method which is expected to make the value of C.I. good. We also compare this method with Harker's method [2] by a computational experiment.

## 2. Characteristic Polynomials

In this section, we investigate the characteristic polynomial of a positive reciprocal matrix. When the matrix is consistent in the sense of Saaty, the characteristic polynomial has a considerably simple form. Surprisingly, we will show that only one coefficient of the polynomial completely determines whether the matrix is consistent or not.

We begin by definitions.

**Definition 1** Let  $A = (a_{ij})$  be an  $n \times n$  real matrix.

1.  $A$  is said to be *positive* provided that  $a_{ij} > 0$  for all  $i, j = 1, \dots, n$ .
2.  $A$  is said to be *reciprocal* provided that  $a_{ij} = 1/a_{ji}$  for all  $i, j = 1, \dots, n$ .

Hence, a positive reciprocal matrix has a form such as

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

**Definition 2** Let  $A = (a_{ij})$  be an  $n \times n$  positive reciprocal matrix.  $A$  is said to be *consistent* (in Saaty's sense) provided that  $a_{ij}a_{jk} = a_{ik}$  for all  $i, j, k = 1, \dots, n$ .

We consider the characteristic polynomial of a positive reciprocal matrix  $A$ :

$$P_A(\lambda) := \det(\lambda E - A).$$

We will show several properties of  $P_A(\lambda)$  in connection with the consistency of  $A$ . From the general theory of the characteristic polynomials, we know that

$$P_A(\lambda) = \lambda^n - (\text{trace}A)\lambda^{n-1} + \cdots + (-1)^n \det A.$$

Since  $A$  is positive and reciprocal, the diagonal elements of  $A$  are all 1. Thus we have

$$P_A(\lambda) = \lambda^n - n\lambda^{n-1} + \cdots + (-1)^n \det A.$$

When  $A$  is consistent,  $P_A(\lambda)$  has a considerably simple form. The following result was implicitly suggested in [8, Chapter 2, Theorem 2.2].

**Proposition 1** Let  $A$  be an  $n \times n$  positive reciprocal matrix. Then  $A$  is consistent if and only if

$$P_A(\lambda) = \lambda^n - n\lambda^{n-1}.$$

*Proof:* If  $P_A(\lambda) = \lambda^n - n\lambda^{n-1}$ , then the characteristic equation  $P_A(\lambda) = 0$  has solutions  $\lambda = 0, n$ . Hence, the maximum eigenvalue of  $A$  is  $n$ , which is the case  $A$  is consistent (see [10]).

Contrary, let  $A$  be consistent. Then we have

$$\begin{aligned}
 \det(A - \lambda E) &= \det \begin{pmatrix} 1 - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 - \lambda \end{pmatrix} \\
 &= a_{21} \cdots a_{n1} \det \begin{pmatrix} 1 - \lambda & a_{12} & \cdots & a_{1n} \\ 1 & a_{12}(1 - \lambda) & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{12} & \cdots & a_{1n}(1 - \lambda) \end{pmatrix} \\
 &= \det \begin{pmatrix} 1 - \lambda & 1 & \cdots & 1 \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix} \\
 &= \det \begin{pmatrix} n - \lambda & n - \lambda & \cdots & n - \lambda \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix} \\
 &= (n - \lambda) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix} \\
 &= (n - \lambda) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda \end{pmatrix} \\
 &= (n - \lambda)(-\lambda)^{n-1}.
 \end{aligned}$$

This completes the proof. □

**Remark 1** In the proof of Proposition 1, we only use the relation

$$a_{1j}a_{jk} = a_{1k}, \quad \text{for all } j, k = 1, \dots, n. \quad (2.1)$$

This means that it suffices to show (2.1) to verify the consistency of  $A$ . This fact is directly proven as follows. Let  $A$  satisfy (2.1), then for all  $i, j, k = 1, \dots, n$  we have

$$a_{ij}a_{jk} = (a_{1j}a_{i1})(a_{1k} a_{j1}) = a_{1k}a_{i1} = a_{ik}.$$

In the context of AHP, a pairwise comparison matrix is desirable to be consistent so that  $\lambda_{\max} = n$ . If the comparison is inconsistent, the additional terms of the degree less than  $n - 1$  appear and they give an effect to make the value  $\lambda_{\max}$  larger than  $n$ . We will examine the additional terms in detail. To calculate the coefficients of the additional terms, we use the following 'Frame method' [3].

**[Frame method]**

Let  $A$  be an  $n \times n$  matrix. Define  $c_k$  and  $A_k$ ,  $k = 0, \dots, n$  as follows.

**Step 1** Set  $c_0 = 1$ ,  $A_0 = E$ ,  $k = 1$ . Go to Step 2.

**Step 2** Compute

$$c_k = -\frac{\text{trace}(AA_{k-1})}{k},$$

$$A_k = AA_{k-1} + c_k E.$$

Go to Step 3.

**Step 3** If  $k = n$  then stop, or else set  $k = k + 1$  and go to Step 2.

Then the resulting numbers  $c_k$ ,  $k = 1, \dots, n$  are the coefficients of the characteristic polynomial of  $A$ . That is, we have

$$P_A(\lambda) = \lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n.$$

Using the frame method, we calculate  $c_2$  and  $c_3$ . For positive reciprocal matrices, the coefficient  $c_2$  always vanishes.

**Proposition 2** *If  $A$  is positive and reciprocal, then  $c_2 = 0$ .*

*Proof:* Applying the frame method iteratively, we have

$$k = 0, \quad c_0 = 1, \quad A_0 = E,$$

$$k = 1, \quad c_1 = -\text{trace}(AA_0) = -\text{trace}A = -n,$$

$$A_1 = AA_0 + c_1 E = A - nE,$$

$$k = 2, \quad c_2 = -\frac{1}{2}\text{trace}(AA_1) = -\frac{1}{2}\{\text{trace}A^2 - n \cdot \text{trace}A\}.$$

Since  $(i, i)$ -component of  $A^2$  is  $\sum_{j=1}^n a_{ij}a_{ji} = n$ , we have

$$c_2 = -\frac{1}{2}(n^2 - n \cdot n) = 0.$$

□

The consistency of the  $2 \times 2$  positive reciprocal matrix follows from the propositions above.

**Corollary 1** *Any  $2 \times 2$  positive reciprocal matrix is consistent.*

*Proof:* It is obvious from Propositions 1 and 2. □

The coefficient  $c_3$  of the degree  $n - 3$  is rather simple so that we can treat it easily.

**Proposition 3** *Let  $n \geq 3$ . If  $A$  is a positive reciprocal  $n \times n$  matrix, then*

$$c_3 = 2 \binom{n}{3} - \sum_{i < j < k} \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right)$$

$$= \sum_{i < j < k} \left\{ 2 - \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) \right\}.$$

*Proof:* By the frame method, we have

$$A_2 = AA_1 + c_2E = AA_1 = A^2 - nA$$

and

$$c_3 = -\frac{1}{3}\text{trace}(AA_2) = -\frac{1}{3}\text{trace}(A^3 - nA^2) = -\frac{1}{3}\{\text{trace}A^3 - n^3\}.$$

Hence, we get the required result from the following lemma.  $\square$

**Lemma 1** *Let  $n \geq 3$ . If  $A$  is a positive reciprocal  $n \times n$  matrix, then*

$$\text{trace}A^3 = 3 \sum_{i < j < k} \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) - 6 \binom{n}{3} + n^3.$$

*Proof:* We prove by induction on  $n$  of the size of the matrix  $A$ .

1° Let  $n = 3$ . Then  $P_A(\lambda) = \lambda^3 - 3\lambda^2 - \det A$ . Hence, by the frame method, we have

$$-\det A = c_3 = -\frac{1}{3}\text{trace}(A^3 - 3A^2) = -\frac{1}{3}\text{trace}A^3 + 3^2.$$

Direct calculation shows that

$$\det A = \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - 2.$$

Thus the required result holds when  $n = 3$ .

2° Let  $A$  be an  $(n+1) \times (n+1)$  positive reciprocal matrix. Then we have

$$\begin{aligned} \text{trace}A^3 &= \sum_{i,j,k=1}^{n+1} a_{ij}a_{jk}a_{ki} \\ &= \sum_{i,j,k=1}^n a_{ij}a_{jk}a_{ki} \end{aligned} \tag{2.2}$$

$$\begin{aligned} &+ \sum_{j,k=1}^n a_{n+1j}a_{jk}a_{kn+1} + \sum_{i,k=1}^n a_{in+1}a_{n+1k}a_{ki} + \sum_{j,i=1}^n a_{ij}a_{jn+1}a_{n+1i} \\ &+ \sum_{k=1}^n a_{n+1n+1}a_{n+1k}a_{kn+1} + \sum_{i=1}^n a_{in+1}a_{n+1n+1}a_{n+1i} \\ &+ \sum_{j=1}^n a_{n+1j}a_{jn+1}a_{n+1n+1} \\ &+ a_{n+1n+1}a_{n+1n+1}a_{n+1n+1} \end{aligned} \tag{2.3}$$

By the assumption of the induction, we have

$$(2.2) = 3 \sum_{1 \leq i < j < k \leq n} \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) - 6 \binom{n}{3} + n^3.$$

By the direct calculation, we have

$$\begin{aligned} (2.3) &= \sum_{j \neq k}^n a_{n+1j}a_{jk}a_{kn+1} + \sum_{k \neq i}^n a_{in+1}a_{n+1k}a_{ki} + \sum_{i \neq j}^n a_{ij}a_{jn+1}a_{n+1i} \\ &+ \sum_{j=1}^n a_{n+1j}a_{jj}a_{jn+1} + \sum_{k=1}^n a_{kn+1}a_{n+1k}a_{kk} + \sum_{i=1}^n a_{ii}a_{in+1}a_{n+1i} \\ &= 3 \sum_{1 \leq i < j \leq n} \left( \frac{a_{ij}a_{jn+1}}{a_{in+1}} + \frac{a_{in+1}}{a_{ij}a_{jn+1}} \right) + 3n. \end{aligned}$$

Hence, we have

$$\begin{aligned} \text{trace}A^3 &= 3 \sum_{1 \leq i < j < k \leq n+1} \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) - 6 \binom{n}{3} + n^3 + 3n + 3n + 1 \\ &= 3 \sum_{1 \leq i < j < k \leq n+1} \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) - 6 \binom{n+1}{3} + (n+1)^3. \end{aligned}$$

□

**Corollary 2** *Let  $n \geq 3$ . If  $A$  is a positive reciprocal  $n \times n$  matrix, then  $c_3 \leq 0$ .*

*Proof:* From the well-known inequality between arithmetic and geometric means, we have

$$\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \geq 2.$$

□

Once we obtain the exact formula indicated in Proposition 3, we can judge the consistency of  $A$  only from the coefficient  $c_3$ .

**Theorem 1** *Let  $n \geq 3$  and  $A$  be a positive reciprocal  $n \times n$  matrix. Then  $A$  is consistent if and only if  $c_3 = 0$ .*

*Proof:* From Proposition 3, we know

$$c_3 = \sum_{i < j < k} \left\{ 2 - \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) \right\}.$$

Hence, it is obvious that  $c_3 = 0$  when  $A$  is consistent. On the contrary, let  $c_3 = 0$ . As was mentioned in the proof of Corollary 2,

$$\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \geq 2.$$

Hence,  $c_3 = 0$  is equivalent to

$$\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} = 2, \quad \text{for all } i < j < k.$$

It occurs if and only if

$$\frac{a_{ij}a_{jk}}{a_{ik}} = \frac{a_{ik}}{a_{ij}a_{jk}}, \quad \text{for all } i < j < k.$$

By positivity of  $A$ , we have

$$a_{ij}a_{jk} = a_{ik}, \quad \text{for all } i < j < k.$$

Setting  $i = 1$ , in particular, we have

$$a_{1j}a_{jk} = a_{1k}, \quad \text{for all } j < k.$$

For  $k < j$ , we have

$$a_{1j}a_{jk} = a_{1j} \frac{1}{a_{kj}} = a_{1j} \frac{1}{a_{1k}a_{kj}} a_{1k} = a_{1j} \frac{1}{a_{1j}} a_{1k} = a_{1k}.$$

As we mentioned in Remark 1, these relations assure the consistency of  $A$ . □

### 3. Estimating a Missing Datum of Incomplete Matrices

The purpose of this section is to present a method for estimating a missing datum of an incomplete matrix. The method is based on the results of the previous section. We also discuss the comparison with Harker's method [2].

#### 3.1. Proposed Method

We consider an incomplete pairwise comparison matrix which has *one* missing entry. Without loss of generality, we may assume the missing entry is  $(1, n)$ -component (hence, so is  $(n, 1)$ -component) of the matrix. We denote the value of the missing entry by  $x$  and the matrix by  $A(x)$ . Hence, the incomplete matrix treated here has the following form:

$$A(x) = \begin{pmatrix} 1 & a_{12} & \cdots & x \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/x & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

Denote the largest eigenvalue of  $A(x)$  by  $\lambda_{\max}(x)$ . We are content if we can solve the following optimization problem:

$$\min_x \lambda_{\max}(x). \quad (3.1)$$

Several studies are devoted to solve the minimization problem of the largest eigenvalue (see [4, 5] and references therein). Though these studies have been succeeded much, they are not applicable to our problem. It is because they mainly treat *symmetric* matrices.

Instead of solving (3.1) exactly, we propose a heuristic method which is based on the results of the previous section. It is expected to make the value  $\lambda_{\max}(x)$  good. Let  $c_3(x)$  be a coefficient of  $\lambda^{n-3}$  of the characteristic polynomial of  $A(x)$ . From Corollary 2 and Theorem 1, we know  $c_3(x) \leq 0$  and  $A(x)$  becomes consistent if we are able to make  $c_3(x) = 0$ . Hence, it is expected that the consistency of  $A(x)$  gets better as  $c_3(x)$  gets closer to 0. Based on the idea, we propose the following method. Consider

$$\max_x c_3(x), \quad (3.2)$$

instead of (3.1). Then we proceed:

#### [Proposed method]

**Step 1** Find  $x_0$  of the solution of the problem (3.2).

**Step 2** Calculate the largest eigenvalue of  $A(x_0)$  and its associate eigenvector.

**Step 3** Normalize the eigenvector into a priority weight vector.

From Proposition 3,

$$c_3 = \sum_{i < j < k} \left\{ 2 - \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) \right\}.$$

Thus the problem (3.2) is equivalent to

$$\min_x \sum_{j=2}^{n-1} a_{1j}a_{jn} \frac{1}{x} + \sum_{j=2}^{n-1} \frac{1}{a_{1j}a_{jn}} x. \quad (3.3)$$

The solution  $x_0$  of the problem (3.3) is easily verified such that

$$x_0 = \sqrt{\left( \sum_{j=2}^{n-1} a_{1j}a_{jn} \right) / \left( \sum_{j=2}^{n-1} \frac{1}{a_{1j}a_{jn}} \right)}.$$

**Remark 2** (i) When  $n = 3$ , it is easily seen that this method yields  $c_3(x_0) = 0$  so that the resulting matrix  $A(x_0)$  is consistent. (ii) When the missing entry is  $(i, j)$ -component, it is obvious that  $x_0$  should be

$$x_0 = \sqrt{\frac{\sum_{\substack{k=1 \\ k \neq i, j}}^n a_{ik}a_{kj}}{\sum_{\substack{k=1 \\ k \neq i, j}}^n \frac{1}{a_{ik}a_{kj}}}}.$$

### 3.2. Harker's Method

To estimate missing data, Harker proposed another heuristic method [2, 9]. His method is based on the following idea. If  $(i, j)$ -component is missing, put the artificial value  $w_i/w_j$  into the vacant component to construct a complete reciprocal matrix  $A(w)$ . Then consider the eigensystem problem:

$$A(w) \cdot w = \lambda w.$$

For example, let

$$A = \begin{pmatrix} 1 & 2 & \square \\ 1/2 & 1 & 2 \\ \square & 1/2 & 1 \end{pmatrix},$$

where  $\square$  is a missing entry. Then

$$A(w) \cdot w = \begin{pmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

which gives  $\tilde{A} \cdot w$  after simplification with

$$\tilde{A} = \begin{pmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix}.$$

Formally, Harker's method is written as follows. Given incomplete matrix  $A = (a_{ij})$ , define the corresponding *derived reciprocal matrix*  $\tilde{A} = (\tilde{a}_{ij})$  by

$$\tilde{a}_{ij} = \begin{cases} 1 + m_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j \text{ and } (i, j)\text{-component is missing,} \\ a_{ij}, & \text{otherwise,} \end{cases}$$

where  $m_i$  denotes the number of missing components in the  $i$ -th row.

#### [Harker's method]

**Step 1** Construct a derived reciprocal matrix  $\tilde{A}$  of  $A(x)$ .

**Step 2** Calculate the largest eigenvalue  $\tilde{\lambda}_{\max}$  of  $\tilde{A}$  and its associate eigenvector.

**Step 3** Normalize the eigenvector into a priority weight vector.

Hence, if we apply Harker's method to  $A(x)$  having  $(1, n)$ -missing component, we have

$$\tilde{A} = \begin{pmatrix} 2 & a_{12} & \cdots & 0 \\ a_{21} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & 2 \end{pmatrix}.$$

We discuss the comparison between Harker's method and our proposed method. The comparison is especially stressed on the values of  $\tilde{\lambda}_{\max}$  and  $\lambda_{\max}(x_0)$ . In some cases, both methods are coincident.



**Theorem 2** Let  $(1, n)$ -component is the only missing entry of  $A$ . If

$$a_{1j}a_{jn} = a_{1k}a_{kn}, \quad \text{for all } j, k = 2, \dots, n-1,$$

then Harker's method and our method are coincident.

*Proof:* If we set  $\bar{a}_{1n} = x_0 = \sqrt{(\sum_{j=2}^{n-1} a_{1j}a_{jn}) / (\sum_{j=2}^{n-1} \frac{1}{a_{1j}a_{jn}})}$ , then we have

$$\bar{a}_{1n} = a_{1j}a_{jn}, \quad \text{for all } j = 2, \dots, n-1. \quad (3.4)$$

Set  $\bar{a}_{n1} = 1/\bar{a}_{1n}$  and

$$\bar{A} = A(x_0) = \begin{pmatrix} 1 & a_{12} & \cdots & \bar{a}_{1n} \\ a_{21} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & a_{n2} & \cdots & 1 \end{pmatrix}.$$

Let  $\lambda_{\max}(x_0)$  be the largest eigenvalue of our eigensystem problem  $\bar{A} \cdot w = \lambda w$  and  $w$  be an associated eigenvector of it. We show that  $\lambda_{\max}(x_0)$  and  $w$  are also a solution of Harker's eigensystem problem  $\tilde{A} \cdot w = \lambda w$ . Since

$$\tilde{A} - \bar{A} = \begin{pmatrix} 1 & \cdots & -\bar{a}_{1n} \\ \vdots & O & \vdots \\ -\bar{a}_{n1} & \cdots & 1 \end{pmatrix},$$

it is enough to show that  $w_1 - \bar{a}_{1n}w_n = 0$ . Since  $\bar{A} \cdot w = \lambda_{\max}(x_0)w$ , we have

$$w_1 + a_{12}w_2 + \cdots + \bar{a}_{1n}w_n = \lambda_{\max}(x_0)w_1, \quad (3.5)$$

$$\bar{a}_{n1}w_1 + a_{n2}w_2 + \cdots + w_n = \lambda_{\max}(x_0)w_n. \quad (3.6)$$

From (3.4), we know  $\bar{a}_{1n} \cdot a_{nj} = a_{1j}$ . Hence, we subtract (3.6)  $\times \bar{a}_{1n}$  from (3.5) to obtain  $\lambda_{\max}(x_0)(w_1 - \bar{a}_{1n}w_n) = 0$ . It is known that  $\lambda_{\max}(x_0) \geq n$  [10], so we have  $w_1 - \bar{a}_{1n}w_n = 0$ . Hence, we see  $\lambda_{\max}(x_0)$  is also an eigenvalue of Harker's eigensystem problem.

We can verify  $\lambda_{\max}(x_0) = \tilde{\lambda}_{\max}$  as follows. Let  $v$  be a left eigenvector of  $v \cdot \tilde{A} = \lambda v$  associated to  $\tilde{\lambda}_{\max}$ . We note that, by the Perron-Frobenius Theorem [3], we may assume  $v, w > 0$ . Then we have

$$\tilde{\lambda}_{\max}v \cdot w = v \cdot \tilde{A} \cdot w = v \cdot \lambda_{\max}(x_0)w = \lambda_{\max}(x_0)v \cdot w,$$

which means  $\tilde{\lambda}_{\max} = \lambda_{\max}(x_0)$ . If we remark that  $\tilde{\lambda}_{\max} \geq n$  generally holds [2], we can show in the same manner that  $\tilde{\lambda}_{\max}$  is also the maximum eigenvalue of our eigensystem.  $\square$

### 3.3. Computational Experiment

In this subsection, we give a computational experiment which compares the values of  $\lambda_{\max}(x_0)$  and  $\tilde{\lambda}_{\max}$ . In the context of AHP, the maximum eigenvalue is desirable to be as small as possible so that the comparison would be near consistent. So, it is natural to consider that  $\lambda_{\max}(x_0)$  measures the performance of our method and  $\tilde{\lambda}_{\max}$  measures the performance of Harker's method, respectively. Hence, we judge a method is good if it shows less value than the other. The experiment is designed as follows to judge which method reveals a good performance.

**[Experiment]**

**Step 1** Randomize  $a_{ij}$  with values  $1, \dots, 9$  and their reciprocal for  $i < j$  except  $a_{1n}$ .

**Step 2** Construct  $A(x_0)$  and  $\tilde{A}$ .

**Step 3** Compare  $\lambda_{\max}(x_0)$  and  $\tilde{\lambda}_{\max}$ .

To calculate eigenvalues, we use the power method with stopping criterion  $\epsilon = 10^{-8}$ . As for the power method and its computer implementation including stopping criterion, consult [10]. A program is written in GNU C. Under 10,000 trials for each size of  $n = 4, \dots, 15$ , we obtained the following table.

Table 1: Results

$n$	4	5	6	7	8	9
# of win	9,998	7,727	6,756	6,079	5,647	5,397
$n$	10	11	12	13	14	15
# of win	5,204	5,085	5,181	5,003	5,033	4,984

In the table, ‘# of win’ counts how many times the situation  $\lambda_{\max}(x_0) < \tilde{\lambda}_{\max}$  occurs. That is the case when our method reveals better performance than Harker’s.

From the result, we may say that our method is better than Harker’s except  $n = 15$ . Particularly it shows a marked trend when the size of the matrix is not large. The competition gets fierce when the size gets large. In the practical use of AHP, the size of the pairwise comparison matrix is recommended to be limited under  $7 \pm 2$  [7, Chapter 3], so our method seems to be effective in practice. As for the difference between  $\tilde{\lambda}_{\max}$  and  $\lambda_{\max}(x_0)$ , the cases that satisfy  $(\tilde{\lambda}_{\max} - \lambda_{\max}(x_0))/\tilde{\lambda}_{\max} > 0.01$  occurred 1,560 times when  $n = 4$ , 505 times when  $n = 5$ , 12 times when  $n = 6$  and no times when  $n > 6$ .

#### 4. Conclusion

This paper has primarily presented the relationships between the consistency and the characteristic polynomials of a positive reciprocal matrix. The most interesting fact is concerning with the coefficient  $c_3$  of the degree  $n - 3$  of the characteristic polynomial. The pairwise comparison is consistent if and only if its coefficient  $c_3 = 0$ . This property is not able to expected for coefficients of the degree other than  $n - 3$ . For example, let us consider a  $4 \times 4$  positive reciprocal matrix  $A$ . In this case, the characteristic polynomial turns to be

$$P_A(\lambda) = \lambda^4 - 4\lambda^3 + c_3\lambda + \det A.$$

Let us consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1/2 & 1 \end{pmatrix},$$

which is obviously inconsistent. It is clear that  $\det A = 0$ . We see that the nullity of the coefficient  $c_4$  cannot assure the consistency.

Secondary, we have proposed a method of estimating a missing datum for the incomplete matrices based on this interesting result above. In the present paper, we have only proposed the method for one missing datum. This is a drawback of our method comparing with

Harker's method. Harker's method has an advantage in the sense that it can treat matrices of many missing data. When there exists more than two missing data, the difficulty of our method stems mainly from solving

$$\max_{x_1, x_2, \dots} c_3(x_1, x_2, \dots). \quad (4.1)$$

In some cases, we can solve (4.1) easily. For example let the  $(i, j)$  and  $(k, l)$  components be missing. If  $i, j, k, l$  are different each other, it is obvious that the solution of (4.1) has the form represented by the formula in Remark 2 (ii).

Since  $c_3(x_1, x_2, \dots)$  is a posynomial, (4.1) can be converted to an equivalent minimization problem of a differentiable convex function [1]. Unfortunately, this problem often has infinitely many solutions. So the new question arises. Which solution is the best candidate? To answer the question, we feel that we should take account in the coefficients other than  $c_3$ . The problem addressed here is left for the future research.

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Shunsuke Shiraishi

Faculty of Economics, Toyama University  
Toyama 930-8555, Japan  
e-mail: shira@eco.toyama-u.ac.jp

Tsuneshi Obata

Faculty of Engineering, Oita University  
Oita 870-1192, Japan  
e-mail: obata@csis.oita-u.ac.jp

Motomasa Daigo

Faculty of Economics, Doshisha University  
Kamigyou-ku, Kyoto 602-8580, Japan  
e-mail: mdaigo@mail.doshisha.ac.jp