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A NOTE ON A THEOREM OF CONTINUUM OF ZERO POINTS

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Abstract By developing an algorithm, Herings, Talman and Yang [6] recently proved the following interesting and deep theorem: A correspondence ζ from the n-dimensional unit cube U^n to the n-dimensional Euclidean space R^n has a continuum of zero points containing the origin and the vector of all-ones if the correspondence satisfies certain conditions. In this note we give an alternative proof of the theorem in the case where the correspondence $\zeta : U^n \to R^n$ is single-valued.

1. Introduction

Since the appearance of the celebrated Brouwer's fixed point theorem, various fixed point theorems have been established. However, most of them only guarantee the existence of a single fixed point, and, as far as we are aware of, there are few existence results for multiple fixed points. Herings, Talman and Yang [6] recently demonstrated, by developing a simplicial algorithm, the following theorem (Theorem 4.3 of [6]) in a constructive manner: there exists a connected set of zero-points of a correspondence $\zeta : U^n \to R^n$ containing the origin and the vector of all-ones if the correspondence satisfies some suitable conditions, where R^n is the n-dimensional Euclidean space and U^n is the unit cube of R^n, i.e., $U^n := \{ x \mid x \in R^n;\ 0 \le x_i \le 1 \text{ for all } i = 1, 2, \ldots, n \}$. This theorem is interesting and deep, because it implies Brouwer's fixed point theorem as a special case. For the above theorem, Herings and Talman [5] provided an alternative existence proof. In this note we present a new and intuitively understandable proof of this theorem based on a theorem of Browder [1]. Our framework also gives a natural path-following interpretation of the algorithm of Herings, Talman and Yang [6], which is based on the three decades of significant development of fixed point computation, e.g., Scarf [9], Eaves [3], van der Laan and Talman [8], and Kojima and Yamamoto [7].

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we present the main result. In Section 4, we give an example as a geometric interpretation of the theorem.

2. Preliminaries

We denote the set of all real numbers by R, the set of integers $\{1, 2, \ldots, n\}$ by I_n. For any $x, y \in R^n$, $x \ge y$ means $x_i \ge y_i$ for $i \in I_n$, $x > y$ means $x \ge y$ with some $j \in I_n$ such that $x_j > y_j$, and $x \gg y$ means $x_i > y_i$ for $i \in I_n$. We write $\mathbf{0} := (0, 0, \ldots, 0)^\top$, $\mathbf{e} := (1, 1, \ldots, 1)^\top$ and $\mathbf{e}^i := (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)^\top$ for $i \in I_n$ and $R^n_{++} := \{ x \mid x \in R^n;\ x \gg \mathbf{0} \}$. For a subset A of R^n, ∂A denotes the boundary of A in R^n. A function $P : R^n \to U^n$ is said to be the *orthogonal projection* from R^n onto U^n if

$$P(x) := \text{argmin}\{\, \|x-y\| \mid y \in U^n \,\}, \tag{2.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. For a subset X of R^n and a function $h : X \times [0,1] \to X$, we define a subset C_h of $X \times [0,1]$ as follows.

$$C_h := \{\, (x,t) \mid (x,t) \in X \times [0,1];\ x = h(x,t) \,\}. \tag{2.2}$$

Let $f : U^n \to R^n$ be a continuous function and let X be an open convex subset of R^n containing U^n. We define a function $\theta : X \to R^n$ by

$$\theta(x) := P(x) + f(P(x)). \tag{2.3}$$

Since the orthogonal projection P is a retraction onto U^n, θ is continuous on X and also $\theta(x) = x + f(x)$ for $x \in U^n$.

For each $t \in [0,1]$, we define the subset $\Omega(t)$ of U^n and a retraction $r_t : R^n \to \Omega(t)$ as follows.

$$\Omega(t) := \{\, x \mid x \in U^n;\ \textstyle\sum_{i=1}^n x_i = nt \,\} \tag{2.4}$$

$$r_t(x) := \text{argmin}\{\, \|x-y\| \mid y \in \Omega(t) \,\}. \tag{2.5}$$

The topological space Y is said to be *connected* if it is not the union of two or more nonempty disjoint closed sets. A subset of Y is called connected if it is connected as a subspace of Y. It is well known that the connectedness is a topologically invariant property. In particular, the continuous image of a connected set remains connected. For a subset Z of R^n and a point $x \in Z$, the *connected component of* x *in* Z is the union of all connected subsets of Z containing x. A subset of Z is simply called a *connected component* of Z if it is a connected component of some point in Z.

3. Main Result

To prove the main theorem we require the following theorem, which is a special case of Theorem 2 in Browder [1].

Theorem 1 *Let X be an open convex subset of R^n and let $h : X \times [0,1] \to X$ be a continuous function. Suppose there exists a compact subset K of X such that $h(X \times [0,1]) \subset K$. Then there exists a connected component D of C_h defined by (2.2) such that both of $D \cap (X \times \{0\})$ and $D \cap (X \times \{1\})$ are nonempty.*

In this theorem, $X \times \{t\}$ is homeomorphic to X for each fixed $t \in [0,1]$. Then the function $h(\cdot, t)$ has a fixed point in X for each $t \in [0,1]$ by Brouwer's fixed point theorem. We here introduce two lemmas necessary to prove the main theorem.

Lemma 2 *For the retraction $r_t : R^n \to \Omega(t)$ of (2.5), it holds that $\|r_t(x) - r_t(x')\| \leq \|x - x'\|$ for $x, x' \in R^n$.*

The proof is straightforward and will be omitted. The following is Corollary 8.1 of Hogan [4]. Let 2^{R^n} denote the power set of R^n.

Lemma 3 *Let $F : R \to 2^{R^n}$ be a point-to-set map and let $g : R \times R^n \to R$ be a real-valued function. Let the point-to-set map $k : R \to 2^{R^n}$ be defined as*

$$k(t) := \mathit{argmin}\{\, g(t,y) \mid y \in F(t) \,\}. \tag{3.1}$$

Suppose F is continuous at $\bar{t}$ in the sense of point-to-set map, g is continuous on $\{\bar{t}\} \times F(\bar{t})$, k is nonempty and uniformly compact near $\bar{t}$, and $k(\bar{t})$ is a singleton. Then k is continuous at $\bar{t}$.

Lemma 2 and Lemma 3 yield the following theorem.

Theorem 4 *Let X be an open convex subset of R^n containing U^n and let $f : U^n \to R^n$ be a continuous function. Let the orthogonal projection $P : R^n \to U^n$, the function $\theta : X \to R^n$, the set $\Omega(t)$ and the retraction r_t be defined by (2.1), (2.3), (2.4) and (2.5), respectively. Then the function $h : X \times [0,1] \to U^n$ defined by*

$$h(x,t) := r_t(\theta(x)) \tag{3.2}$$

is continuous on $X \times [0,1]$.

Proof: First we show the continuity of h with respect to t. For a fixed $x \in X$, let $F : [0,1] \to 2^{U^n}$ and $g : [0,1] \times U^n \to R$ be defined by

$$F(t) := \Omega(t) \text{ and } g(t,y) := \|\theta(x) - y\|^2.$$

Also let $k : [0,1] \to 2^{U^n}$ be defined by (3.1) for these F and g. Since $F(t)$ is a compact convex set and $g(t,y)$ is a strictly convex function in y, $k(t)$ is a singleton. The other conditions of Lemma 3 can be easily checked, and since $h(x,t) = k(t)$, h is continuous with respect to t.

Next we show the continuity of h with respect to $(x,t) \in X \times [0,1]$. Let $\varepsilon > 0$ be given. By the continuity of θ on X, there exists $\delta_1 > 0$ such that for $x' \in X$

$$\|\theta(x') - \theta(x)\| < \varepsilon/2 \text{ if } \|x' - x\| < \delta_1. \tag{3.3}$$

Using the continuity of h in t, we can choose $\delta_2 > 0$ such that for $t' \in [0,1]$

$$\|h(x,t') - h(x,t)\| < \varepsilon/2 \text{ if } |t' - t| < \delta_2. \tag{3.4}$$

Put $\delta := \min\{\delta_1, \delta_2\}$. For any $(x',t') \in X \times [0,1]$ with $\|x' - x\| < \delta$ and $|t' - t| < \delta$, using (3.3), (3.4) and Lemma 2, we see

$$\begin{aligned} \|h(x',t') - h(x,t)\| &\leq \|h(x',t') - h(x,t')\| + \|h(x,t') - h(x,t)\| \\ &< \|r_{t'}(\theta(x')) - r_{t'}(\theta(x))\| + \varepsilon/2 \\ &\leq \|\theta(x') - \theta(x)\| + \varepsilon/2 < \varepsilon. \end{aligned}$$

Therefore h is continuous at (x,t). □

Now we introduce the assumption of f in [6], which plays the crucial role for the existence of a continuum of zero points.

Assumption 5 *The function $f : U^n \to R^n$ satisfies*

(a) f is continuous,

(b) for any $x \in \partial U^n$,

$$f_i(x) \begin{cases} \geq 0 & \text{if } x_i = 0 \\ \leq 0 & \text{if } x_i = 1, \end{cases}$$

(c) for each $x \in U^n$, there is $p(x) \in R^n_{++}$ such that $p(x)^\top f(x) = 0$.

By Z_f we denote the set of zero points of f, i.e.,

$$Z_f := \{ x \mid x \in U^n;\ f(x) = \mathbf{0} \}.$$

Remark: According to Assumption 5 (c), either $f(x) \geq \mathbf{0}$ or $f(x) \leq \mathbf{0}$ implies $x \in Z_f$.

Now we are ready to prove the existence theorem Theorem 4.3 of Herings, Talman and Yang [6]. It should be noted that their theorem is more general and holds for point-to-set maps.

Theorem 6 *Let $f : U^n \to R^n$ be a function satisfying Assumption 5. Then the set Z_f contains a connected component containing* $\mathbf{0}$ *and* $\mathbf{e}$.

To prove the theorem, we first prove the following lemma.

Lemma 7 *Let $f : U^n \to R^n$ be the function satisfying Assumption 5 and let $h : X \times [0,1] \to U^n$ be defined by (3.2). Then for $(x,t) \in X \times [0,1]$, $(x,t) \in C_h$ if and only if $x \in Z_f \cap \Omega(t)$.*

Proof: To prove the "only if" part let (x,t) be a point in C_h. Then we see

$$x = h(x,t) = r_t(\theta(x)) \in \Omega(t).$$

Then $P(x) = x$ and $x = r_t(x + f(x))$. Now we only need to show that $x \in Z_f$. We define $\tilde{f}$, h and g_i for $i \in I_{2n}$ as follows:

$$\tilde{f}(y) := \frac{1}{2}\|y - (x + f(x))\|^2, \quad h(y) := \sum_{j=1}^{n} y_j - nt$$

$$g_i(y) := \begin{cases} -y_i & \text{if } i \in I_n \\ y_{i-n} - 1 & \text{if } i \in I_{2n} \setminus I_n, \end{cases}$$

and consider the minimization problem:

$$\left|\begin{array}{ll} \text{minimize} & \tilde{f}(y) \\ \text{subject to} & y \in X;\ h(y) = 0;\ g_i(y) \le 0 \text{ for } i \in I_{2n}. \end{array}\right.$$

Note that the feasible region of this problem is exactly $\Omega(t)$. Let $J := \{\, i \mid i \in I_{2n};\ g_i(y) = 0 \,\}$, which is the union of $J_0 := \{\, i \mid i \in I_n;\ y_i = 0 \,\}$ and $J_1 := \{\, i \mid i \in I_n;\ y_i = 1 \,\}$. Due to the linearity of constraints, it is readily seen that the problem satisfies a suitable constraint qualification, e.g., Abadie's constraint qualification. Therefore we obtain the necessary condition at a solution x that

$$f(x) = \lambda_0 \mathbf{e} + \sum_{i \in J_0} \lambda_i(-\mathbf{e}^i) + \sum_{i \in J_1} \lambda_i \mathbf{e}^i$$

for some $\lambda_0 \in R$ and $\lambda_i \ge 0$ for $i \in J$. If $x \notin \partial U^n$, then $J_0 \cup J_1 = \emptyset$ and this condition reduces to $f(x) = \lambda_0 \mathbf{e}$. According to Assumption 5 (c), we obtain $\lambda_0 = 0$, and hence $x \in Z_f$.

We then assume $x \in \partial U^n$ and consider the following three cases.

Case1: $J_0 \ne \emptyset$ and $J_1 \ne \emptyset$.

By Assumption 5 (b), for any $i \in J_0$ and $i' \in J_1$, we have

$$\lambda_0 - \lambda_i \ge 0 \text{ and } \lambda_0 + \lambda_{i'} \le 0,$$

so that $\lambda_0 = 0$. Then $\lambda_i = 0$ and $\lambda_{i'} = 0$, and hence $x \in Z_f$.

Case 2: $J_0 \ne \emptyset$ but $J_1 = \emptyset$

By Assumption 5 (b), we have $\lambda_0 - \lambda_i \ge 0$ for $i \in J_0$. Then $\lambda_0 \ge 0$ and

$$f(x) = \lambda_0 \mathbf{e} + \sum_{i \in J_0} \lambda_i(-\mathbf{e}^i) \ge \mathbf{0}.$$

Applying Remark, we obtain $x \in Z_f$.

Case 3: $J_0 = \emptyset$ but $J_1 \ne \emptyset$

By Assumption 5 (b), we have $\lambda_0 + \lambda_i \le 0$ for $i \in J_1$. Then $\lambda_0 \le 0$ and

$$f(x) = \lambda_0 \mathbf{e} + \sum_{i \in J_1} \lambda_i \mathbf{e}^i \le \mathbf{0}.$$

Again applying Remark, we obtain $x \in Z_f$.

Next we prove the "if" part. Suppose $x \in Z_f \cap \Omega(t)$, then $P(x) = x$ and $f(x) = \mathbf{0}$. Thus

$$\theta(x) = P(x) + f(P(x)) = x$$

and

$$h(x,t) = r_t(\theta(x)) = r_t(x) = x,$$

because $x \in \Omega(t)$. Therefore $(x,t) \in C_h$ and the proof is completed. □

Now we give the proof of Theorem 6.

Proof of Theorem 6

Note that the function $h : X \times [0,1] \to U^n$ of (3.2) is continuous by Theorem 4. According to Theorem 1 there exists a connected subset D of C_h such that both $D \cap (U^n \times \{0\})$ and $D \cap (U^n \times \{1\})$ are nonvacant. Suppose $(x,0),(x',1) \in D$, then we have

$$x = h(x,0) = r_0(\theta(x)) = \mathbf{0}$$

and

$$x' = h(x',1) = r_1(\theta(x')) = \mathbf{e}.$$

Then $(\mathbf{0},0),(\mathbf{e},1) \in D$. Let $P_x : X \times [0,1] \to X$ be the projection onto the first coordinate. Since P_x is continuous and D is a connected subset of C_h, we obtain a connected set $P_x(D) \subset Z_f$ which contains two points

$$\mathbf{0} = P_x((\mathbf{0},0)) \text{ and } \mathbf{e} = P_x((\mathbf{e},1)).$$

Now the proof is completed. □

4. Example

In this section, we give an illustrative example as a geometric interpretation of Theorem 6. Let $f : U^2 \to R^2$ be defined by

$$f(x) := ((x_2^\mu - x_1^\nu)(1+x_1), (x_1^\nu - x_2^\mu)(1+x_2))^\top,$$

where μ and ν are natural numbers. For each μ and ν, the continuity of f is clear. We easily see that

$$f_1(0,x_2) = x_2^\mu \geq 0;\ f_2(x_1,0) = x_1^\mu \geq 0$$

and

$$f_1(1,x_2) = 2(x_2^\mu - 1) \leq 0;\ f_2(x_1,1) = 2(x_1^\nu - 1) \leq 0.$$

Next, for each $x \in U^2$ let $p(x) = (1+x_2, 1+x_1)^\top \in R^2_{++}$, then $p(x)^\top f(x) = 0$. Hence Assumption 5 (a), (b) and (c) are satisfied. It is clear to see that Z_f contains a connected component

$$S := \{\, (t, t^{\frac{\nu}{\mu}})^\top \mid t \in [0,1] \,\}$$

connecting $\mathbf{0}$ and $\mathbf{e}$. The component S of this example can have the following three different shapes depending on the values of μ and ν:

(i) If $\mu = \nu$, $S = \{\, (t,t) \mid t \in [0,1] \,\}$, which is the diagonal set of U^2.

(ii) If $\mu > \nu$, S is an arc linking two corners above the diagonal set.

(iii) If $\mu < \nu$, S is an arc linking two corners below the diagonal set.

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