RANDOMIZED LOOK STRATEGY FOR A MOVING TARGET WHEN A SEARCH PATH IS GIVEN

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Abstract   This paper investigates the search problem for a moving target when a search path is given in advance. The searcher's strategy is represented by a randomized look strategy, that is, the probability with which he looks in his current position at each discrete time. The searcher knows the probabilistic law about which one of some options the target selects as his path. If the detection of the target occurs by the looking, the searcher gains some value but must expend some cost for the looking. The searcher wants to determine his optimal look strategy in order to maximize the expected reward which is defined as the expected value minus the expected cost. He can randomize his look strategy by the probability that he looks in his current position. We prove the NP-completeness of this problem and propose a dynamic programming method to give an optimal solution which becomes the bang-bang control in the result. We derive some characteristics of the optimal look strategy and analyze them by some numerical examples.

1.   Introduction

Since the 1970's, Pollock[1], Hellman[2], Iida[3], Dobbie[4], Kan[5] and other researchers have studied optimal search problems for a moving target. Brown[6] proposed an efficient algorithm to find an optimal allocation of continuously divisible search resources maximizing the detection probability of the moving target and Washburn[7] generalized this method and called it the FAB algorithm. Stromquist and Stone[8] made a great contribution to purify this kind of optimizing search problems mathematically. In many studies, the criterion of the optimization was the detection probability. Iida and Hozaki[9] investigated an optimal solution based on the reward criterion by which the searcher could consider the search from the view point of the cost-performance.

In these mathematical models, the searcher is allowed to move or allocate search resources anywhere he likes. In recent years, the search problem in which the searcher path is constrained has been discussed by some researchers. In this problem, called the path constrained search problem(the PCSP), the ability of the searcher's movement is taken into consideration in the sense that the next positions he can move to are limited depending on his current position. This assumption looks natural in some realistic models. Eagle[10] proposed a solution method based on the dynamic programming to give an optimal search path of maximizing the detection probability. Stewart[11] proposed an approximate solution procedure using the branch and bound method for this problem. Eagle and Yee[12] extended the Stewart's method to an exact method, which performs better than the previous dynamic programming method. Under the criterion of the detection probability, the search must be conducted at all times because the search operation costs nothing. If we need to consider the search cost in the operation, a non-search strategy could be optimal from the view point of cost-performance. Hohzaki and Iida[13] proposed a branch and bound method to give an

optimal strategy which consists of two strategies, the route strategy and the look strategy, for the path constrained search problem with the reward criterion. Nakai[14] studied an optimal look strategy of maximizing the detection probability in a simple model which is the same as ours.

In this paper, we consider one of the simplest models of the PCSPs with the reward criterion in the discrete search time period T. The searcher is given a search path in advance. A target takes one of some options of his paths with a certain probability law which is known to the searcher. The detection of the target at time t brings some value V_t to the searcher and then the search operation terminates. The detection at time t could occur only by the searcher's looking in his current position, which expends search cost c_t. The searcher is interested in determining his optimal look strategy of maximizing the expected reward during T which is defined as the expected value minus the expected cost. The look strategy is randomized and represented by the look probabilities $\{\varphi(t), t \in T\}$ so that the searcher looks in his current position with probability $\varphi(t)$ and does not look with probability $1 - \varphi(t)$ at time t.

In the next section, we describe the mathematical model of our problem. The NP-completeness of our problem is proved in Section 3. A dynamic programming approach to an optimal look strategy and the discussion about some characteristics of the optimal solution are developed in Section 4. By some numerical examples, we analyze the sensitivity of system parameters and verify the validity of theorems obtained in this paper.

2. Modeling of Search Problem

In this section, we state some assumptions of a search problem and construct the mathematical model. Assumptions of the search problem are as follows.

(1) A searcher moves on a search space in discrete time $T = \{1, \cdots, T\}$. A path is given to the searcher in advance. A target moves on the same search space. The target has some options of paths to move along. The whole set of paths is denoted by Ω. When path $\omega \in \Omega$ comes across the search path on the way is denoted by $\omega(t)$. That is, if the path ω and the search path cross each other at time $t \in T$, $\omega(t) = 1$ and otherwise $\omega(t) = 0$. The probability $\pi_0(\omega)$ of the target's selecting path ω is guessed for $\omega \in \Omega$ and known to the searcher.

(2) The searcher can searches (looks) into his current position to detect the target, if necessary. If the target is there when the searcher looks at time t, the searcher can detect the target with probability p_t. Events of the detection at each time are independent of each other.

(3) At time t, the searcher gains non-negative value V_t on the detection of the target but loses non-negative cost c_t as the looking cost.

(4) The searcher determines his randomized look strategy $\{\varphi(t), t \in T\}$ at each time and does not change it during the search operation. $\varphi(t)$ denotes the probability with which the searcher looks in his current position at time t and is between 0 and 1 of course.

(5) The searcher wants to find his optimal look strategy of maximizing the expected reward which is defined as the expected value minus the expected cost.

3. NP-Completeness

We simplify our model so that the look strategy is not randomized but includes only two options of looking or non-looking at each time. By assumptions stated in the previous

section, the searcher comes cross the set of the target paths $\Omega_t = \{\omega \in \Omega | \omega(t) = 1\}$ at time t. Now we consider a decision problem $MRPGP(\Omega_1, \Omega_2, \cdots, \Omega_T, m; R)$ which answers whether or not the searcher can gain a certain reward R by m looks at most, where MRPGP is the abbreviation for Maximum Reward Problem with Given Search Path. When we specify value V_t by a positive constant V and make cost c_t be zero, $MRPGP(\Omega_1, \cdots, \Omega_T, m; R)$ becomes a decision problem concerning the detection probability, $MDPGP(\Omega_1, \cdots, \Omega_T, m; P)$ which answers whether or not the detection probability $P = R/V$ can be obtained by m looks at most, where MDPGP is the abbreviation for Maximum Detection Probability Problem with Given Search Path.

Let us prove the NP-completeness of $MRPGP(\cdot)$ by that of $MDPGP(\cdot)$. Now we consider an instance of $MDPGP(\Omega_1, \cdots, \Omega_T, m; P)$ as follows.

(1) In the search operation, the perfect detection is performed, that is, $p_t = 1$, $t \in T$. When the searcher looks in the cell and the target is there, the detection occurs with certainty.

(2) The probability of the target path selection is positive, that is, $\pi_0(\omega) > 0$ for any $\omega \in \Omega$.

(3) The searcher can come across the target on any path sometime. That is, $\bigcup_{t=1}^{T} \Omega_t = \Omega$.

(4) P equals one and m is less than T.

Since the searcher can detect the target coming across with certainty, this instance is equivalent to the question about whether or not the searcher can select m looks or m subsets exhaustively covering the whole path set Ω from the target path subsets $\Omega_1, \Omega_2, \cdots, \Omega_T$. This problem is nothing but so-called the set covering problem $SCP(\Omega_1, \cdots, \Omega_T; m)[15]$ which is known to be NP-complete. Therefore, the problem $MRPGP(\Omega_1, \cdots, \Omega_T, m; R)$ includes the set covering problem as a special case. The input length of the former problem is $|\Omega| T + \sum_{\omega \in \Omega} \lceil \log(1/\pi_0(\omega)) \rceil + \sum_{t=1}^{T} \lceil \log(1/p_t) \rceil + \sum_{t=1}^{T} \lceil \log V_t \rceil + \sum_{t=1}^{T} \lceil c_t \rceil + m + \lceil \log R \rceil$ while that of the latter problem is $|\Omega| T + m$ under an adequate encoding scheme. We can easily prove that the set covering problem $SCP(\Omega_1, \cdots, \Omega_T; m)$ is polynomial transformable to the above instance of $MRPGP(\Omega_1, \cdots, \Omega_T, m; R)$ with parameters $V_t = 1$, $c_t = 0$, $p_t = 1$, too.

Theorem 1 $MDPGP(\Omega_1, \cdots, \Omega_T, m; P)$ and $MRPGP(\Omega_1, \cdots, \Omega_T, m; R)$ are both NP-complete.

4. Formulation of the Problem and Optimal Look Strategy

Here we formulate an objective function of the problem by the randomized look strategy of the searcher $\{\varphi(t), t = 1, \cdots, T\}$. At the first, let us consider the search operation during time period $[t, T] \subseteq [1, T]$. Given that the target takes path $\omega \in \Omega$, the detection probability of the target $P_t^\tau(\varphi, \omega)$ and the cumulative search cost $C_t^\tau(\varphi)$ during $[t, \tau]$ is given for $t \leq \tau \leq T$ by

$$P_t^\tau(\varphi, \omega) = 1 - \prod_{\xi=t}^{\tau}(1 - \omega(\xi)\varphi(\xi)p_\xi) , \tag{4.1}$$

$$C_t^\tau(\varphi) = \sum_{\xi=t}^{\tau} c_\xi \varphi(\xi) . \tag{4.2}$$

The target is detected at time τ with the probability $P_t^\tau(\varphi, \omega) - P_t^{\tau-1}(\varphi, \omega)$ and if so, the searcher gains the value V_τ but has expended the cumulative search cost $C_t^\tau(\varphi)$. If the detection does not occur until T, the searcher only loses the search cost $C_t^T(\varphi)$. Therefore, the expected reward during $[t, T]$ is given by

$$R_t^T(\varphi,\omega) = \sum_{\tau=t}^{T}(V_\tau - C_t^\tau(\varphi))(P_t^\tau(\varphi,\omega) - P_t^{\tau-1}(\varphi,\omega)) - C_t^T(\varphi)(1 - P_t^T(\varphi,\omega)) \qquad (4.3)$$

where $P_t^{t-1}(\varphi,\omega)$ is defined to be zero. Because the non-detection probability $1 - P_t^\tau(\varphi,\omega)$ during $[t,\tau]$ is the probability that there occurs no detection at time t and no detection during $[t+1,\tau]$, we have $1 - P_t^\tau(\varphi,\omega) = (1 - P_t^t(\varphi,\omega))(1 - P_{t+1}^\tau(\varphi,\omega))$, which is verified from Eq. (4.1) too. By this equation, Eq. (4.2) and $P_t^t(\varphi,\omega) = \omega(t)\varphi(t)p_t$, we can transform the reward $R_t^T(\varphi,\omega)$ as follows.

$$R_t^T(\varphi,\omega)$$

$$= (V_t - C_t^t(\varphi))P_t^t(\varphi,\omega) + (1 - P_t^t(\varphi,\omega)) \sum_{\tau=t+1}^{T} (V_\tau - C_t^\tau(\varphi))(P_{t+1}^\tau(\varphi,\omega) - P_{t+1}^{\tau-1}(\varphi,\omega))$$

$$-(1 - P_t^t(\varphi,\omega))C_t^T(\varphi)(1 - P_{t+1}^T(\varphi,\omega))$$

$$= (V_t - C_t^t(\varphi))P_t^t(\varphi,\omega) + (1 - P_t^t(\varphi,\omega))$$

$$\times \left\{ \sum_{\tau=t+1}^{T} (V_\tau - C_{t+1}^\tau(\varphi))(P_{t+1}^\tau(\varphi,\omega) - P_{t+1}^{\tau-1}(\varphi,\omega)) - C_{t+1}^T(\varphi)(1 - P_{t+1}^T(\varphi,\omega)) - c_t\varphi(t) \right\}$$

$$= V_t\omega(t)\varphi(t)p_t - c_t\varphi(t) + R_{t+1}^T(\varphi,\omega) - \omega(t)\varphi(t)p_t R_{t+1}^T(\varphi,\omega) . \qquad (4.4)$$

When the searcher executes the looking at time t and it fails to detect the target, the posterior probability of the target's selecting path ω is calculated by

$$\Lambda_t\pi(\omega) \equiv Pr\{the\ target\ selects\ path\ \omega \mid non-detection\ occurs\ at\ time\ t\ \}$$

$$= \frac{\pi(\omega)(1 - p_t\omega(t))}{1 - p_t \sum_{\omega'\in\Omega_t} \pi(\omega')} . \qquad (4.5)$$

Assuming the probability law of target path selection is $\{\pi(\omega)\}$ at the beginning of time t, the whole expected reward $\widetilde{R}_t^T(\varphi,\pi)$ during $[t,T]$ over all target paths is given by weighting $R_t^T(\varphi,\omega)$ with probability $\pi(\omega)$.

$$\widetilde{R}_t^T(\varphi,\pi) = \sum_{\omega\in\Omega} \pi(\omega)R_t^T(\varphi,\omega) . \qquad (4.6)$$

Noting that $\sum_{\omega\in\Omega} \pi(\omega)\omega(t) = \sum_{\omega\in\Omega_t} \pi(\omega)$, by Eqs. (4.4) and (4.5), $\widetilde{R}_t^T(\varphi,\pi)$ can be transformed as follows.

$$\widetilde{R}_t^T(\varphi,\pi)$$

$$= V_t\varphi(t)p_t \sum_{\omega\in\Omega_t} \pi(\omega) - c_t\varphi(t) + \widetilde{R}_{t+1}^T(\varphi,\pi)$$

$$-\varphi(t) \sum_{\omega\in\Omega} \left\{ \pi(\omega) - \left(1 - p_t \sum_{\omega'\in\Omega_t} \pi(\omega')\right) \Lambda_t\pi(\omega) \right\} R_{t+1}^T(\varphi,\omega) \qquad (4.7)$$

$$= (1 - \varphi(t))\widetilde{R}_{t+1}^T(\varphi,\pi) + \varphi(t) \left\{ p_t V_t \sum_{\omega\in\Omega_t} \pi(\omega) - c_t + \left(1 - p_t \sum_{\omega\in\Omega_t} \pi(\omega)\right) \widetilde{R}_{t+1}^T(\varphi,\Lambda_t\pi) \right\} .$$

During time period $[t,T]$, an optimal look strategy of the searcher gives the maximum of the objective function $\widetilde{R}_t^T(\varphi,\pi)$ to the target's path selecting probability $\{\pi(\omega)\}$. Let denote the maximum by $f_t(\pi)$. By using Eq. (4.7), we can partition the optimal value $f_t(\pi)$ as follows.

$$f_t(\pi) = \max_{\{\varphi(t),\cdots,\varphi(T)\}} \widetilde{R}_t^T(\varphi,\pi)$$

$$\le \max_{\varphi(t)} \left[(1 - \varphi(t)) \max_{\{\varphi(t+1),\cdots,\varphi(T)\}} \widetilde{R}_{t+1}^T(\varphi,\pi) \right.$$

$$\left. + \varphi(t) \left\{ p_t V_t \sum_{\omega\in\Omega_t} \pi(\omega) - c_t + \left(1 - p_t \sum_{\omega\in\Omega_t} \pi(\omega)\right) \max_{\{\varphi(t+1),\cdots,\varphi(T)\}} \widetilde{R}_{t+1}^T(\varphi,\Lambda_t\pi) \right\} \right]$$

$$= \max_{0 \le \varphi(t) \le 1} \{(1 - \varphi(t))f_{t+1}(\pi) + \varphi(t)g_t(\pi)\} = \begin{cases} f_{t+1}(\pi), & if \ f_{t+1}(\pi) \ge g_t(\pi) \\ g_t(\pi), & if \ f_{t+1}(\pi) < g_t(\pi) \end{cases} , \quad (4.8)$$

where $g_t(\pi)$ is defined by

$$g_t(\pi) \equiv p_t V_t \sum_{\omega \in \Omega_t} \pi(\omega) - c_t + \left(1 - p_t \sum_{\omega \in \Omega_t} \pi(\omega)\right) f_{t+1}(\Lambda_t \pi) . \quad (4.9)$$

As known from the above transformation, an optimal look strategy during $[t, T]$ is constructed by adding $\{\varphi(t) = 0\}$ to the optimal strategy of $f_{t+1}(\pi)$ during $[t + 1, T]$ or adding $\{\varphi(t) = 1\}$ to the optimal strategy of $g_t(\pi)$ during $[t + 1, T]$. If we move time point t from T to 1, we can obtain the optimal strategy during the whole time period. This characteristic reminds us the dynamic programming technique which usually takes the time flow in reverse order. From now, we use the dynamic programming for convenience and reverse the time flow of the problem so that time points $t = 1, 2, \cdots, T$ are interchanged with $k = T, T - 1, \cdots, 1$, respectively, where term k means the residual time points up to terminating time T. We redefine notation $\omega(t), p_t, V_t, c_t, \varphi(t), \Omega_t, \Lambda_t \pi, f_t(\pi)$ and $g_t(\pi)$ defined on time t in the normal time flow by $\omega(k), p_k, V_k, c_k, \varphi(k), \Omega_k, \Lambda_k \pi, f_k(\pi)$ and $g_k(\pi)$ on term k in the reversed time flow, respectively.

We reconstruct our optimizing problem in the fasion of the dynamic programming because it is easy and comprehensible. If the looking is executed at term k, it gives the searcher the possible gain V_k with the detection probability $p_k \sum_{\omega \in \Omega_k} \pi(\omega)$ and inevitable expense c_k. The failure of the detection transforms the path selection probability π into the posterior probability $\Lambda_k \pi$ and the term changes to $k - 1$. If the looking is not executed, the searcher is given no gain and no cost with no change of π. We already have the function $f_k(\pi)$ defined on term k which is the maximum expected reward by an optimal look strategy since term k given that the path selection probability at the beginning of term k is π. This objective function satisfies the following recursive equation, which is clear from the above explanation or Eq. (4.8). We represent one of the optimal look strategies by $\{\varphi^*(k), k = T, \cdots, 1\}$.

$$\begin{aligned} f_k(\pi) &= \max_{0 \le \varphi(k) \le 1} \Big[(1 - \varphi(k))f_{k-1}(\pi) \\ &\quad + \varphi(k) \left\{ p_k V_k \sum_{\omega \in \Omega_k} \pi(\omega) - c_k + \left(1 - p_k \sum_{\omega \in \Omega_k} \pi(\omega)\right) f_{k-1}(\Lambda_k \pi) \right\} \Big] \\ &= \max_{0 \le \varphi(k) \le 1} [(1 - \varphi(k))f_{k-1}(\pi) + \varphi(k)g_k(\pi)] \\ &= \begin{cases} f_{k-1}(\pi), & if \ f_{k-1}(\pi) \ge g_k(\pi); \ \varphi^*(k) = 0 \ \ in \ this \ case \\ g_k(\pi), & if \ f_{k-1}(\pi) < g_k(\pi); \ \varphi^*(k) = 1 \ \ in \ this \ case \end{cases} \\ &= \max \{f_{k-1}(\pi), g_k(\pi)\} , \end{aligned} \quad (4.10)$$

where

$$g_k(\pi) = p_k V_k \sum_{\omega \in \Omega_k} \pi(\omega) - c_k + \left(1 - p_k \sum_{\omega \in \Omega_k} \pi(\omega)\right) f_{k-1}(\Lambda_k \pi). \quad (4.11)$$

An initial condition is given by

$$f_0(\pi) = 0 \ \ for \ any \ \pi . \quad (4.12)$$

We may proceed the recursive estimation (4.10) from $k = 1$ upto T and find $f_T(\pi_0)$. In Eq.(4.10), $f_{k-1}(\pi)$ and $g_k(\pi)$ indicate the possible maximum rewards in the case of the non-looking and the looking at term k, respectively. By this dynamic programming approach, we can elucidate some characteristics of the optimal strategy.

Theorem 2 *The optimal strategy has the following characteristics.*

(i)   *There is always an optimal look strategy of being the bang-bang control, that is, $\varphi^*(k) = 1$ or 0 for any term k.*

(ii) $f_k(\pi)$ *is non-decreasing function for term* k*, that is,* $0 \leq f_{k-1}(\pi) \leq f_k(\pi)$*.*

(iii) *If the searcher does not hit any target paths at term* k*, there is an optimal strategy so that the searcher does not look at term* k*.*

(iv) *If there is a certain* $n \leq T$ *and*

$$p_k V_k \leq c_k \quad for \quad k = n, n-1, \cdots, 1 \tag{4.13}$$

holds, there is an optimal strategy so that the searcher finishes the search at term n*, that is,* $\varphi^*(k) = 0, \ k = n, \cdots, 1$*.*

(v) *If* $f_T(\pi) = 0$*, the following holds.*

$$p_k V_k \sum_{\omega \in \Omega_k} \pi(\omega) \leq c_k \quad for \ any \ k \ . \tag{4.14}$$

However this relation is not reversible.

Proof. The characteristic (i) is self-explanatory from Eq. (4.10). The characteristic (ii) is derived from $f_k(\pi) = \max\{f_{k-1}(\pi), g_k(\pi)\} \geq f_{k-1}(\pi)$. If $\Omega_k = \emptyset$, $g_k(\pi) \leq f_{k-1}(\pi)$ from Eq. (4.11) and we have the characteristic (iii). We can verify (iv) as follows. If the conditon (4.13) holds, from Eq.(4.11),

$$g_1(\pi) = p_1 V_1 \sum_{\omega \in \Omega_1} \pi(\omega) - c_1 \leq p_1 V_1 - c_1 \leq 0 \ .$$

Therefore $f_1(\pi) = 0$ for any π from Eq.(4.10). By the mathematical induction, we can show $f_2(\pi) = f_3(\pi) = \cdots = f_n(\pi) = 0$ for any π, that is, $\varphi^*(k) = 0, \ k = 1, \cdots, n$.

If Eq.(4.14) does not hold for some k, we adopt the following look strategy; $\varphi(k) = 1$ and $\varphi(l) = 0$ for any $l \neq k$. The expected reward of this case is given by $p_k V_k \sum_{\omega \in \Omega_k} \pi(\omega) - c_k$ which is positive. This contradicts the optimality of $f_T(\pi) = 0$. Therefore, the characteristic (v) is proved. **Q.E.D.**

Now we define the space of $\{\pi(\omega), \ \omega \in \Omega\}$ by

$$\Pi \equiv \{\pi(\omega), \ \omega \in \Omega \mid \pi(\omega) \geq 0 \ , \ \sum_{\omega \in \Omega} \pi(\omega) = 1\} \ .$$

For any $\pi \in \Pi$, an optimal look strategy and its optimal objective value $f_T(\pi)$ can be calculated. Hence, we can divide the $(|\Omega| - 1)$-dimensional unit simplex Π based on the difference of optimal starategies. We call each of the divided regions the optimal region and call the dividing of the plane the optimal strategy map. Concerning with the function $f_k(\pi)$ and the optimal regions, we can derive the following theorem.

Theorem 3 *The function* $f_k(\pi)$ *is convex for* $\pi \in \Pi$ *and each optimal region is the polyhedral convex set.*

Proof. Let $\pi_1, \pi_2 \in \Pi$ and $0 \leq \lambda \leq 1$. The former fact can be proved by making use of the linearity of the function $\tilde{R}_1^T(\varphi, \pi)$ for π.

$$\begin{aligned}
f_k(\lambda \pi_1 + (1-\lambda)\pi_2) &= \max_\varphi \tilde{R}_{T-k+1}^T(\varphi, \lambda \pi_1 + (1-\lambda)\pi_2) \\
&= \max_\varphi \left\{ \lambda \tilde{R}_{T-k+1}^T(\varphi, \pi_1) + (1-\lambda)\tilde{R}_{T-k+1}^T(\varphi, \pi_2) \right\} \\
&\leq \lambda \max_\varphi \tilde{R}_{T-k+1}^T(\varphi, \pi_1) + (1-\lambda) \max_\varphi \tilde{R}_{T-k+1}^T(\varphi, \pi_2) \\
&= \lambda f_k(\pi_1) + (1-\lambda)f_k(\pi_2) \ .
\end{aligned}$$

Let us prove the latter fact. Assume that two path selection probabilities π_1 and π_2 are elements of the optimal region Π_1 and the optimal look strategy of the region is φ^*. Then the following inequatlity is valid for $0 \leq \lambda \leq 1$ by using the above transformation.

$$f_k(\lambda \pi_1 + (1-\lambda)\pi_2) \leq \lambda \max_\varphi \tilde{R}_{T-k+1}^T(\varphi, \pi_1) + (1-\lambda) \max_\varphi \tilde{R}_{T-k+1}^T(\varphi, \pi_2)$$

$$= \lambda \tilde{R}_{T-k+1}^T(\varphi^*, \pi_1) + (1-\lambda)\tilde{R}_{T-k+1}^T(\varphi^*, \pi_2) = \tilde{R}_{T-k+1}^T(\varphi^*, \lambda \pi_1 + (1-\lambda)\pi_2) \ .$$

Therefore, an optimal look strategy for the combined probability $\lambda\pi_1 + (1-\lambda)\pi_2$ can be φ^* too and $\lambda\pi_1 + (1-\lambda)\pi_2 \in \Pi$ which means the convexity of Π.

Since any optimal region is the convex set and any optimal look strategy becomes the bang-bang control, in other words, the number of optimal strategies is finite and 2^T at most, the plane Π is divided by a finite number of optimal regions. $f_T(\pi)$ is given by finding one of φ maximizing $\sum_\omega \pi(\omega)R_1^T(\varphi,\omega)$ from a finite set of the bang-bang control strategies. Let Π_1 and Π_2 be two optimal regions and φ_1^*, φ_2^* be optimal strategies for Π_1, Π_2, respectively. Since the boundary between two optimal regions Π_1, Π_2 is given by $\sum_\omega \pi(\omega)R_1^T(\varphi_1^*,\omega) = \sum_\omega \pi(\omega)R_1^T(\varphi_2^*,\omega)$, it is apparent that the boundary becomes a hyperplane. Now we complete the proof of the latter fact of this theorem. **Q.E.D.**

5. Numerical Examples

In this section, we take some simple examples and analyze the dependency of optimal strategies on the system parameters. As illustrated in Fig.1, we consider the 3-cells search space, which is indicated by the axis of ordinate and the 3-terms time space which is indicated by the axis of abscissa. A target has 3 paths of option, $\Omega = \{1,2,3\}$, and a searcher is given a search path indicated by a broken line in the figure. Therefore, parameters $\{\omega(k)\}$ are set as follows.

> Path 1: $\omega(3) = 1$, $\omega(2) = 1$, $\omega(1) = 0$,
> Path 2: $\omega(3) = 1$, $\omega(2) = 1$, $\omega(1) = 1$,
> Path 3: $\omega(3) = 0$, $\omega(2) = 1$, $\omega(1) = 0$.

Other system parameters are specified as follows:

> $V_k = V$ *(constant)* , $c_k = c$ *(constant)* .

For simplicity, we denote each probability of target's selecting paths 1, 2, 3 by π_1, π_2, π_3 instead of $\pi(1), \pi(2), \pi(3)$ and $\{\pi_1, \pi_2, \pi_3\}$ by π. We also denote the look strategy at each term of 3, 2, 1 by $\varphi_3, \varphi_2, \varphi_1$ instead of $\varphi(3), \varphi(2), \varphi(1)$ and $\{\varphi_3, \varphi_2, \varphi_1\}$ by φ.
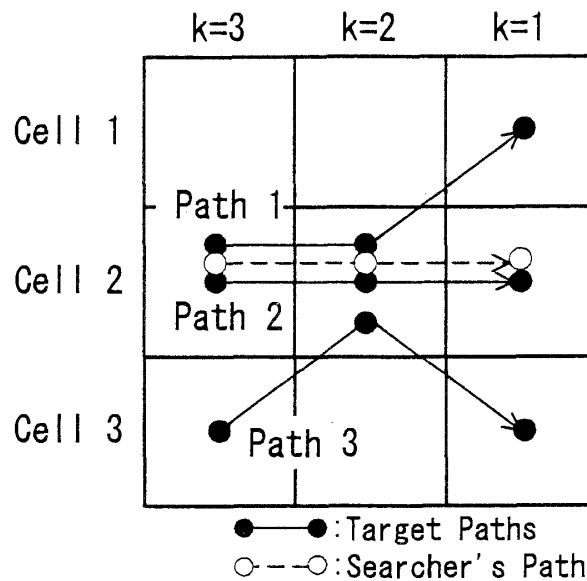

Fig.1 Search space and target paths.

5.1 Sensitivity of target path selecting probability

As a basic example, we take $V = 6, c = 1, p_2 = 1/2$ and estimate the recursive equation (4.10) for $k = 1, 2, 3$. Consequently, the optimal solution is classified in four cases.

(i) Region A: If $\pi_2 < 1/2 - \pi_1$ and $\pi_2 < 1/3$, $\varphi^* = \{0, 1, 0\}$ and $f_3(\pi) = 2$.

(ii) Region B: If $\pi_2 \geq 1/2 - \pi_1$, $\pi_2 > 1 - 4\pi_1$ and $\pi_2 < 1/2 - 1/4\pi_1$, $\varphi^* = \{1, 1, 0\}$ and $f_3(\pi) = 1 + 2(\pi_1 + \pi_2)$.

(iii) Region C: If $\pi_2 \geq 1/3$, $\pi_2 \leq 1 - 4\pi_1$ and $\pi_2 < 2/3 - 3/2\pi_1$, $\varphi^* = \{0, 1, 1\}$ and $f_3(\pi) = 3/2(1 + \pi_2)$.

(iv) Region D: If $\pi_2 \geq 1/2 - 1/4\pi_1$ and $\pi_2 \geq 2/3 - 3/2\pi_1$, $\varphi^* = \{1, 1, 1\}$ and $f_3(\pi) = 9/4\pi_1 + 3\pi_2 + 1/2$.

By Fig.2-a and -b, we illustrate the optimal strategy map and the optimal value function $f_3(\pi)$ on the $\pi_1 - \pi_2$ plane.
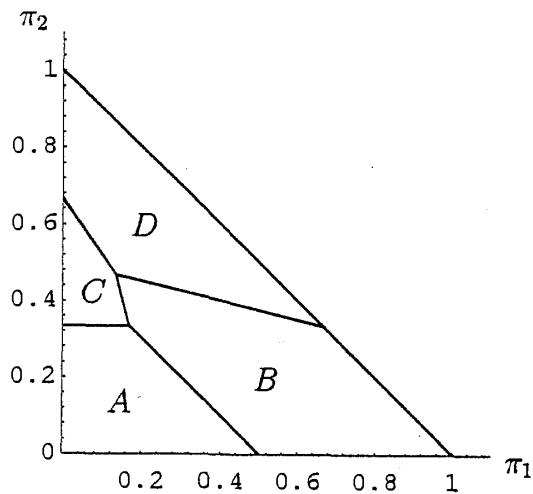

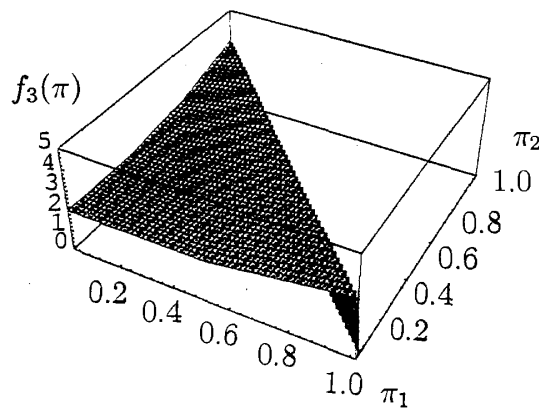Fig.2-a Optimal strategy regions.   Fig.2-b Optimal value of the objective function.

We can explain the change of the optimal strategy corresponding to the target path selecting probability π_1, π_2 and $\pi_3 = 1 - \pi_1 - \pi_2$.

(1) The search path mainly covers target paths 1 and 2. Therefore, the optimal value function $f_3(\pi)$ becomes larger in the case of large $\pi_1 + \pi_2$ than small $\pi_1 + \pi_2$. Especially the search path is coincident with the target path 2, by which larger π_2 is most favorable for the searcher. These facts are shown in Fig.2-b.

(2) Concerning with the optimal strategy map, the following tendency is seen in Fig.2-a. The shape of all the regions is the polytope as stated in Theorem 3 .

   (i) Term 2 at which all target paths gather is most valuable for the search. All optimal strategies have $\varphi_2^* = 1$.

   (ii) Region A: In the case that π_1 and π_2 are small, which means large π_3, only term 2 is to be looked at since the searcher comes across target path 3 only then. The optimal strategy is $\varphi^* = \{0, 1, 0\}$.

   (iii) Region B: In the case of $\pi_1 \approx 1$ and small π_2, term 3 is also valuable by the rendezevous of the searcher and target path 1, as well as term 2. The optimal strategy is $\varphi^* = \{1, 1, 0\}$.

(iv) Region C: In the case of small π_1 and adequate π_2, it would be the good strategy for the searcher to execute the looking at term 1 only if non-detection occurs at term 2. The optimal strategy is $\varphi^* = \{0, 1, 1\}$.

(v) Region D: In the case of large π_2, the searcher must look all the time since the searcher can always cover target path 2.

5.2 Sensitivity of detectability

In the previous basic example, the searcher does not move to other cells except cell 2 and so only the detectability of cell 2, p_2, has effect on the search. We solved the problem with larger detectability $p_2 = 9/10$ and other paramters remaining $V = 6$, $c = 1$. The optimal strategy map is given by Table 1 and Fig.3-a. Figure 3-b presents the optimal values.

Table 1. Optimal strategies

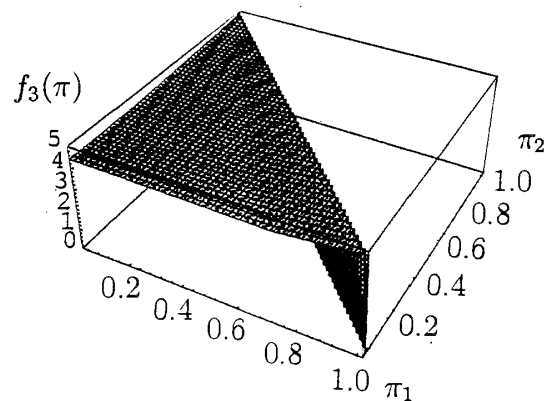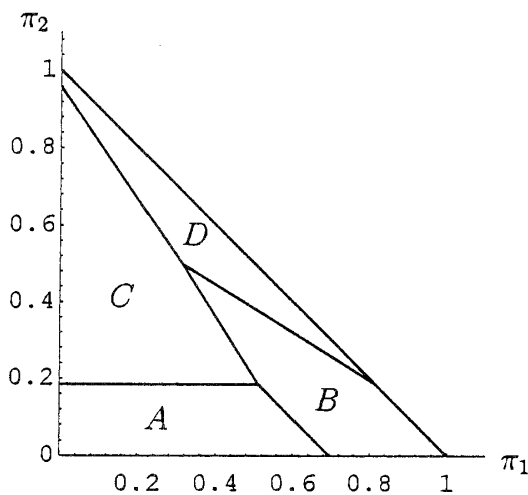| Regions	Conditions	Optimal strategies
A	$\pi_2 \leq 5/27$, $\pi_2 \leq 25/36 - \pi_1$	$\varphi^* = \{0, 1, 0\}$
B	$\pi_2 > 25/36 - \pi_1$, $\pi_2 > 1 - 8/5\pi_1$, $\pi_2 \leq 25/36 - 5/8\pi_1$	$\varphi^* = \{1, 1, 0\}$
C	$\pi_2 > 5/27$, $\pi_2 \leq 1 - 8/5\pi_1$, $\pi_2 \leq 250/261 - 85/58\pi_1$	$\varphi^* = \{0, 1, 1\}$
D	$\pi_2 > 25/36 - 5/8\pi_1$, $\pi_2 > 250/261 - 85/58\pi_1$	$\varphi^* = \{1, 1, 1\}$

Fig.3-a Optimal strategy regions.

Fig.3-b Optimal value of the objective function.

In this case, the searcher can expect the high detection probability at term 2 when all the taget paths gather. Because of that, the looking at term 3 is more wasteful than term 2 and the searcher may wait until term 2 and look then. The optimal region with $\varphi_3^* = 0$, especially region C, has larger area than other regions. By the reason that the most expected reward may be obtained at term 2 with the high detection probability, all the strategies do not make so difference in terms of the optimal reward as seen in Fig.3-b.

5.3 Sensitivity of target value

The optimal strategy map in the case of smaller constant value $V = 3$ and the same parameters $c = 1$, $p_2 = 1/2$ as the basic example is given by Table 2 and Fig.4. The lower target value makes the looking at almost all terms be less attractive and the optimal strategy be $\varphi^* = \{0, 1, 0\}$ almost all everywhere on the map. The search at term 2 yet remains to be

efficient because of the convergence of all the target paths. However, if the value becomes equal to or less than 2, the region of the non-looking strategy $\varphi^* = \{0,0,0\}$ occupies the map because the condition (4.13) of Theorem 2 is satisfied.

Table 2. Optimal strategies

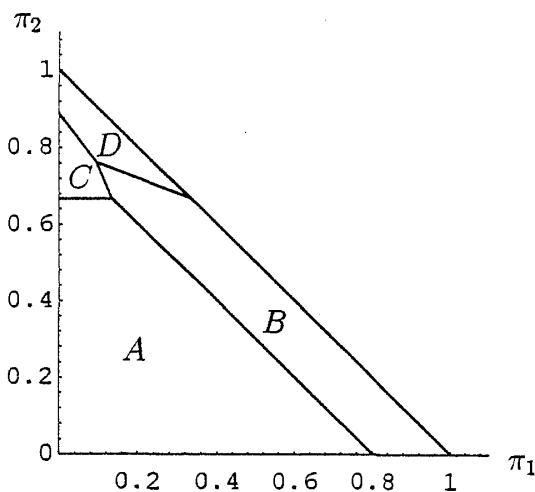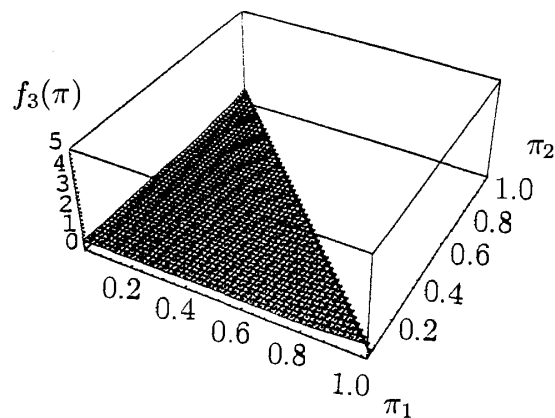| Regions	Conditions	Optimal strategies
A	$\pi_2 \leq 2/3,\ \pi_2 \leq 4/5 - \pi_1$	$\varphi^* = \{0,1,0\}$
B	$\pi_2 > 4/5 - \pi_1,\ \pi_2 > 1 - 5/2\pi_1,\ \pi_2 \leq 4/5 - 2/5\pi_1$	$\varphi^* = \{1,1,0\}$
C	$\pi_2 > 2/3,\ \pi_2 \leq 1 - 5/2\pi_1,\ \pi_2 \leq 8/9 - 4/3\pi_1$	$\varphi^* = \{0,1,1\}$
D	$\pi_2 > 4/5 - 2/5\pi_1,\ \pi_2 > 8/9 - 4/3\pi_1$	$\varphi^* = \{1,1,1\}$

Fig.4-a Optimal strategy regions.

Fig.4-b Optimal value of the objective function.

5.4 Sensitivity of cost

With respect to the sensitivity of cost c in the basic example, we can approximately guess its effect on the optimal strategy map because it will work reversely comparing with target value V.

Here, let us take another example, which is presented by Fig.5. A target is assumed to select path 1 with probability $\pi_0(1) = 2/3$ and path 2 with probability $\pi_0(2) = 1/3$. The target paths set parameters $\{\omega(k)\}$ as follows.

Path 1: $\omega(3) = 0,\ \omega(2) = 1,\ \omega(1) = 1$,
Path 2: $\omega(3) = 1,\ \omega(2) = 1,\ \omega(1) = 0$.

The target value is constant $V = 4$ anywhere and at any time. The detectability of cell 1 and 2 is $p_1 = p_2 = 1/2$. The search cost is assumed to depend only on the search cell and is given by $c_2 = c_a,\ c_1 = c_3 = c_b$. By the dynamic programming procedure (4.10), we obtain the optimal strategy map on the $c_a - c_b$ plane as follows, which is visualized by Fig.6.

(i) Region A: If $c_a \geq 2$ and $c_b \geq 4/3$, $\varphi^* = \{0,0,0\}$ and $f_3(\pi_0) = 0$.
(ii) Region B: If $2c_a - 8/3 \geq c_b,\ 4/3 > c_b \geq 4/5$, $\varphi^* = \{0,0,1\}$ and $f_3(\pi_0) = 4/3 - c_b$.
(iii) Region C: If $3c_a/4 - 1/2 \geq c_b,\ 4/5 > c_b$, and $2c_a - 12/5 \geq c_b$, $\varphi^* = \{1,0,1\}$ and $f_3(\pi_0) = 2 - 11c_b/6$.
(iv) Region D: If $2 \geq c_a$ and $c_b \geq 4/3$, $\varphi^* = \{0,1,0\}$ and $f_3(\pi_0) = 2 - c_a$.

(v)   Region E: If $2c_a - 8/3 < c_b$, $3c_a/4 - 1/2 < c_b$ and $c_b \geq 2c_a/11 + 4/11$, $\varphi^* = \{0, 1, 1\}$ and $f_3(\pi_0) = 8/3 - c_a - 1c_b/2$.

(vi) Region F: If $2c_a - 12/5 < c_b$ and $c_b < 2c_a/11 + 4/11$, $\varphi^* = \{1, 1, 1\}$ and $f_3(\pi_0) = 3 - 5/6c_a - 17c_b/12$.
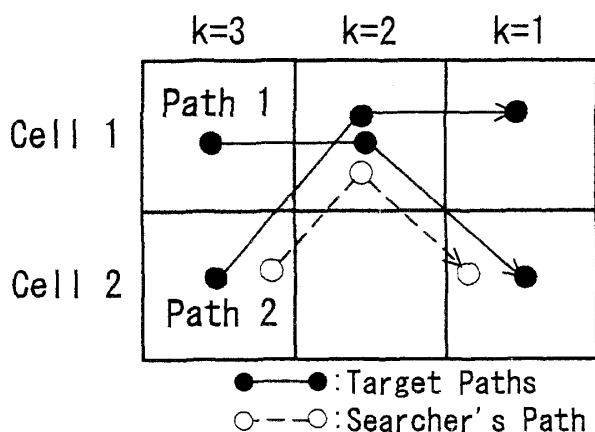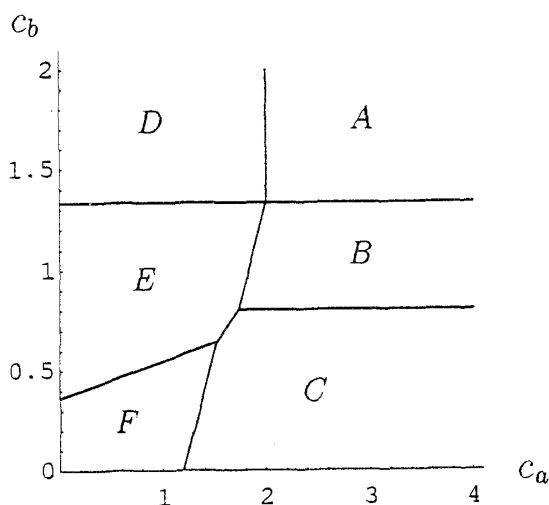

Fig.5 Search space and target paths.

Fig.6 Optimal strategies on the cost plane.

We can explain the allocation of the regions.

- If c_a is large, the looking in cell 1 are expensive and must not be executed, that is, $\varphi_2^* = 0$.

  (i)   Region A: In the case of large c_b, the looking in cell 2 is not favorable too and the optimal strategy becomes $\varphi^* = \{0, 0, 0\}$. The boundaries $c_a = 2$ and $c_b = 4/3$ of this region give the boundaries of the necessary condition (4.14) of Theorem 2 for the non-look strategy.

  (ii) Region B: In the case of adequate c_b, the searching for the target on path 1 becomes a little attractive bacause of its higher path probability and must be done at term 1, $\varphi^* = \{0, 0, 1\}$.

(iii) Region C: In the case of small c_b, path 2 also becomes hopeful for the positive reward, $\varphi^* = \{1, 0, 1\}$.

- If c_a is small, the search in cell 1 at term 2 is desirable, that is, $\varphi_2^* = 1$.

   (iv) Region D: In the case of large c_b, the optimal strategy must be $\varphi^* = \{0, 1, 0\}$ by the same reason as (i).

   (v) Region E: In the case of adequate c_b, the optimal strategy must be $\varphi^* = \{0, 1, 1\}$ by the same reason as (ii).

   (vi) Region F: In the case of small c_b, the optimal strategy must be $\varphi^* = \{1, 1, 1\}$ by the same reason as (iii).

6. Conclusions

In this paper, we deal with a kind of the path constrained search problem where a searcher is given his path in advance and determines an optimal randomized look strategy in order to maximize the expected reward. This problem is a little hard more than it looks and NP-complete, which is proved in this paper. For an optimal solution, we propose a dynamic programming method by which some characteristics of the optimal solution are elucidated: for example, the conditions of the non-looking strategy, the convexity of the optimal value function and so on. Furthermore, we can analyze the sensitivity of system parameters included in the problem while the values of parameters are not specified and remain being variables, by which we can classify the parameter space into some regions according to the optimal strategy.

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