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HOW TO SELECT PAIRED COMPARISONS IN AHP OF INCOMPLETE INFORMATION—STRONGLY REGULAR GRAPH DESIGN

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Abstract It is said that paired comparison is the essence of AHP. But if there are N alternatives and M criteria in a standard AHP, we must compare ${}_MC_2$ pairs for each criterion and ${}_NC_2$ pairs for the set of criteria, and the total number of them becomes up tp ${}_NC_2 \times M + {}_MC_2$. So for rather large M and N it takes much cost and time to get paired comparison data. But even if we have not the whole set S_n of ${}_nC_2$ pairs (let such a case be called incomplete information case), we can estimate the weights based on comparison data in an appropriate subset of S_n by Harker method or Two-stage method [4, 5]. We can use LLS (logarithmic least square) method in AHP analysis, by which we can analyze AHP for incomplete information case. So we can reduce the number of paired comparisons by using incomplete information case. The problem is how to select pairs to be compared in S_n, that is, a design to get data. We propose the strongly regular (SR) design based on strongly regular graphs, and by numerical simulation show that the errors of the estimations by SR designs are smaller than any random designs for almost all cases. Since SR graphs are rather difficult to be constructed, we generalize them to quasi-strongly regular (quasi-SR) graphs, and propose quasi-SR design based on quasi-SR graphs. By simulation we show that quasi-SR designs also give the same good results as the SR designs.

1. Introduction

AHP is an useful method to evaluate alternatives in many decision problems. But if there are N alternatives and M criteria in a standard AHP, we must compare ${}_MC_2$ pairs for each criterion and ${}_NC_2$ pairs for the set of criteria, and the total number of them becomes up to ${}_NC_2 \times M + {}_MC_2$. So for rather large M and N it takes much cost and time to get paired comparison data.

Generally the eigenvalue method of AHP on n objects estimates weights $w_1, w_2, \cdots, w_n$ of objects $1, 2, \cdots, n$, based on ${}_nC_2$ paired comparisons. But even if we have not the whole set S_n of ${}_nC_2$ pairs (let such a case be called incomplete information case), we can estimate $w_1, w_2, \cdots, w_n$ based on comparison data in an appropriate subset of S_n by Harker method or Two-stage method [4, 5]. We can use LLS (logarithmic least square) method in AHP analysis, by which we can analyze AHP for incomplete information case. So we can reduce the number of paired comparisons by using incomplete information case.

The problem is how to select pairs to be compared in S_n, that is, a design to get data. More precisely, for given n, what is the best subset of size m to be selected in S_n, to estimate $w_1, w_2, \cdots, w_n$ based on paired comparisons in the subset.

Taking objects and pairs of objects as points and edges respectively, we have the graph corresponding to paired comparison. A complete graph corresponds to the whole set S_n, and a graph with m edges corresponds to a subset of size m in S_n. Thus our problem is to find what is the best of graphs with m edges to estimate weights of objects.

We propose the strongly regular (SR) design based on strongly regular graphs, and by

numerical simulation show that the errors of the estimations by SR designs are smaller than any random designs for almost all cases in Section 5.

Since SR graph is rather difficult to be constructed, we generalize it to quasi-strongly regular (quasi-SR) graph, and propose quasi-SR design based on quasi-SR graphs. By simulation we show that quasi-SR designs also give the same good results as the SR designs.

There are several methods to construct strongly regular (SR) or quasi-SR graphs. We survey these methods and give new methods based on our own idea and list up them for practical usage in Section 6. For every combination of n and m, we can not always construct SR or quasi-SR graph with n points and m edges. For the case with missing n and m, we can select near n' and m' in the list.

2. Strongly Regular (SR) Graph

Here we treat graphs with undirected edges and without loops nor parallel edges. A complete graph is a graph in which every pair of points is adjacent, and a null graph is a graph without any edges (with only points).

We define the adjacent matrix $\boldsymbol{N} = (N_{ij})$ of graph $\mathbf{G}$ as,

$$N_{ij} = \begin{cases} 1 & \text{if points } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathbf{G}$ has no loops, diagonal elements of $\boldsymbol{N}$ are zero, that is $N_{ii} = 0$, and $\mathbf{G}$ is undirected so $\boldsymbol{N}$ is symmetric. Every element of $\boldsymbol{N}$ of a complete graph is unity except diagonal elements. The adjacent matrix of a null graph is a zero matrix. The complement of graph $\mathbf{G}$ is the graph whose adjacent matrix is $\overline{\boldsymbol{N}} = \boldsymbol{J} - \boldsymbol{I} - \boldsymbol{N}$, where $\boldsymbol{N}$ is the adjacent matrix of $\mathbf{G}$, and $\boldsymbol{J}$ is the all 1 matrix and $\boldsymbol{I}$ is the identity matrix. We call a graph $\mathbf{G}$ *regular* if every point of $\mathbf{G}$ has the same degree (or valency) d. The adjacent matrix $\boldsymbol{N}$ of a regular graph has the following property,

$$\boldsymbol{N}\boldsymbol{J} = \boldsymbol{J}\boldsymbol{N} = d\boldsymbol{J}. \tag{2-1}$$

If a regular graph $\mathbf{G}$ (except complete or null graphs) satisfies the following properties then we call $\mathbf{G}$ *strongly regular* (SR).

1. For any pair of adjacent points p and q, the number of points adjacent to both p and q is λ.
2. For any pair of not adjacent points p and q the number of points adjacent to both p and q is μ.

If an SR graph has n points and the degree is d then we call it (n, d, λ, μ) SR graph. It is clear that if $\mathbf{G}$ is (n, d, λ, μ) SR graph, its complement $\overline{\mathbf{G}}$ is $(n, \overline{d}, \overline{\lambda}, \overline{\mu})$ SR graph where

$$\overline{d} = n - 1 - d, \quad \overline{\lambda} = n - 2d + \mu - 2, \quad \overline{\mu} = n - 2d + \lambda. \tag{2-2}$$

In Fig. 1 we show examples of regular graphs and SR graphs. In (a),(b),(c) of Fig. 1, an SR graph $\mathbf{G}$ and its complement $\overline{\mathbf{G}}$ are shown. Graphs of (d), (e), (f) and (g) are regular but not SR.

If $\boldsymbol{N}$ is an adjacent matrix of an SR graph, then it has the following property [2],

$$\boldsymbol{N}^2 = d\boldsymbol{I} + \lambda\boldsymbol{N} + \mu(\boldsymbol{J} - \boldsymbol{I} - \boldsymbol{N}). \tag{2-3}$$

It is because, for $i \neq j$, $(\boldsymbol{N}^2)_{ij}$ is the number of paths of length 2 from i to j, so if i and j are adjacent ($N_{ij} = 1$ and $(\boldsymbol{J} - \boldsymbol{I} - \boldsymbol{N})_{ij} = 0$) it must be equal to λ, if i and j are not

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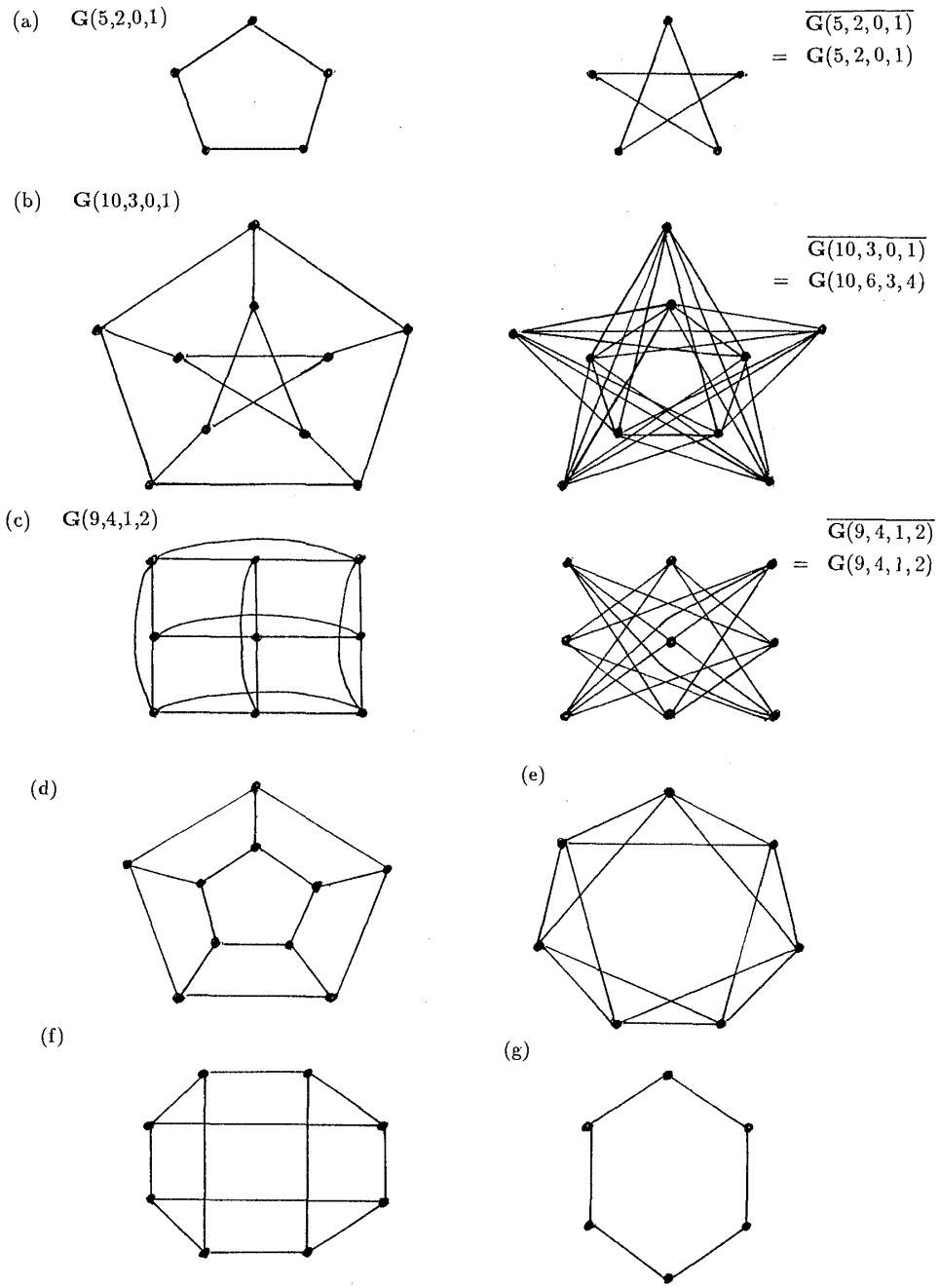

Fig. 1: SR and quasi-SR graphs.

adjacent ($N_{ij} = 0$ and $(\boldsymbol{J} - \boldsymbol{I} - \boldsymbol{N})_{ij} = 1$) it must be equal to μ, and for $i = j$, $(\boldsymbol{N}^2)_{ii}$ is the number of points adjacent to i.

Formulas (2-1) and (2-3) show that the algebra generated by $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{N}$ is linear. In other words any results of adding and multiplying among $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{N}$ are always represented by linear combination of $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{N}$ by (2-3). Indeed, an SR graph could be defined as a regular graph whose adjacent matrix satisfies (2-3). Conversely if $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{N}$ satisfying (2-1) generate a linear algebra on the integral domain, then the graph with adjacent matrix $\boldsymbol{N}$ is SR [2].

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3. Analysis of LLS by SR Design

It is well known [6] that estimates of weights of objects by LLS (Logarithmic Least Square) method are very near to that of eigenvalue method in AHP.

If a_{ij} is the value of paired comparison of object i to j, then we assume the following model,

$$a_{ij} = \frac{w_i}{w_j} e_{ij}, \quad i,j = 1,\cdots,n, \quad (i<j), \tag{3-1}$$

where $e_{ij} > 0$ is error and $a_{ii} = 1$. Taking logarithm of (3-1) we have

$$\log a_{ij} = \log w_i - \log w_j + \log e_{ij}, \quad i,j = 1,\cdots,n, \quad (i<j). \tag{3-2}$$

By minimizing the sum of square of $\log e_{ij}$'s we have LLS estimate $\widehat{w}_i$ of w_i. We assume that $\log e_{ij}$'s are independent random variables and the expectation of $\log e_{ij}$ is zero and its variance is σ^2.

Since $w_1,\cdots,w_n$ are arbitrary by a constant multiple, we can assume $w_1 w_2 \cdots w_n = 1$. Let

$$\overline{w}_i = \log w_i, \quad i = 1,\cdots,n \tag{3-3}$$

then we have

$$\overline{w}_1 + \overline{w}_2 + \cdots + \overline{w}_n = 0. \tag{3-4}$$

Here let us take e as the base.

Example 1 LLS analysis by SR graph $\mathbf{G}(10,3,0,1)$.

Using the SR graph $\mathbf{G}$ in Fig. 2, we have paired comparison data a_{ij} corresponding to edge (i,j) of $\mathbf{G}$.

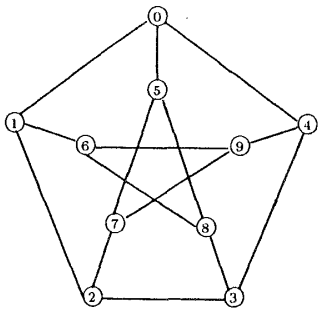

Fig. 2: The graph $\mathbf{G}(10,3,0,1)$.

Setting $\overline{a}_{ij} = \log a_{ij}$ we have a data table like Table 1 and the normal equation like Table 2 where y_i's are linear combination of $\overline{a}_{ij}$'s. Solving the normal equation, we have LLS estimate $\widehat{\overline{w}}_i$ and $\widehat{w}_i = e^{\widehat{\overline{w}}_i}$ $(i = 1,\cdots,n)$.

The precision of estimate $\widehat{\overline{w}}_i$ is measured by its variance $V(\widehat{\overline{w}}_i)$ and this is equal to the value of σ^2 multiplied by i-th diagonal element of inverse matrix $\boldsymbol{M}^{-1}$ of coefficient matrix $\boldsymbol{M}$ of the normal equation. That is

$$V(\widehat{\overline{w}}_i) = M^{ii}\sigma^2, \quad i = 1,\cdots,n, \tag{3-5}$$

where $\boldsymbol{M}^{-1} = (M^{ij})$.

Of course, values of M^{ii}'s are independent of y_i's (or a_{ij}'s), so we can measure goodness of a design by M^{ii} whatever the values of data a_{ij}'s are.

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Table 1: Data table.

	$\overline{w}_0$	$\overline{w}_1$	$\overline{w}_2$	$\overline{w}_3$	$\overline{w}_4$	$\overline{w}_5$	$\overline{w}_6$	$\overline{w}_7$	$\overline{w}_8$	$\overline{w}_9$
$\overline{a}_{01}$	1	−1								
$\overline{a}_{04}$	1				−1					
$\overline{a}_{05}$	1					−1				
$\overline{a}_{12}$		1	−1							
$\overline{a}_{16}$		1					−1			
$\overline{a}_{23}$			1	−1						
$\overline{a}_{27}$			1					−1		
$\overline{a}_{34}$				1	−1					
$\overline{a}_{38}$				1					−1	
$\overline{a}_{49}$					1					−1
$\overline{a}_{57}$						1		−1		
$\overline{a}_{58}$						1			−1	
$\overline{a}_{68}$							1		−1	
$\overline{a}_{69}$							1			−1
$\overline{a}_{79}$								1		−1
0	1	1	1	1	1	1	1	1	1	1

Table 2: Normal equation.

$\overline{w}_0$	$\overline{w}_1$	$\overline{w}_2$	$\overline{w}_3$	$\overline{w}_4$	$\overline{w}_5$	$\overline{w}_6$	$\overline{w}_7$	$\overline{w}_8$	$\overline{w}_9$	
4	0	1	1	0	0	1	1	1	1	y_0
0	4	0	1	1	1	0	1	1	1	y_1
1	0	4	0	1	1	1	0	1	1	y_2
1	1	0	4	0	1	1	1	0	1	y_3
0	1	1	0	4	1	1	1	1	0	y_4
0	1	1	1	1	4	1	0	0	1	y_5
1	0	1	1	1	1	4	1	0	0	y_6
1	1	0	1	1	0	1	4	1	0	y_7
1	1	1	0	1	0	0	1	4	1	y_8
1	1	1	1	0	1	0	0	1	4	y_9

Next we analyze the properties of the coefficient matrix $\boldsymbol{M}$ of normal equation ($\boldsymbol{M}$ is often called *information matrix*) for SR design, and represent $\boldsymbol{M}$ and $\boldsymbol{M}^{-1}$ by linear combinations of $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{N}$.

Theorem 1 *Let $\boldsymbol{M}$ be the information matrix of LLS method for SR design by an SR graph $\mathbf{G}(n, d, \lambda, \mu)$, then*

$$\boldsymbol{M} = d\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{N} \tag{3-6}$$

$$\boldsymbol{M}^{-1} = \alpha\boldsymbol{I} + \beta\boldsymbol{J} + \gamma\boldsymbol{N} \tag{3-7}$$

$$\alpha = \frac{d+\mu-\lambda}{d(d+\mu-\lambda)+(\mu-d)}, \quad \beta = \frac{-2d+\lambda}{n\{d(d+\mu-\lambda)+\mu-d\}}, \quad \gamma = \frac{1}{d(d+\mu-\lambda)+\mu-d}.$$

Proof: [Formula (3-6)]; It is well known in the theory of least square methods that (i, j) element M_{ij} of information matrix $\boldsymbol{M}$ is equal to the inner product of i-th and j-th column vector in data table like Table 1. So M_{ii} is equal to $d+1$, independent of i, because every point has degree d, so there are $d+1$ elements with ± 1 in every column in data table (all elements of the last row in data table is always 1 because of (3-4)). The (i, i) element of the

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right hand side of (3-6) is just equal to $d+1$.

Next if points i and j are not adjacent then the inner product of columns i and j is just 1 (or the rows except the last row, columns i and j do not have common non-zero values), and (i,j) element in the right hand side of (3-6) is just 1 (because $N_{ij}=0$ and $J_{ij}=1$). If points i and j are adjacent then columns i and j have 1 and -1 respectively in the row corresponding to edge (i,j) and both have 1 in the last row, so their inner product is zero while $(d\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{N})_{ij}=J_{ij}-N_{ij}=1-1=0$. So the left hand side of (3-6) completely coincides with the right hand side.

[Formula (3-7)]; As we mentioned above, $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{N}$ generate linear algebra $\boldsymbol{A}$ and $\boldsymbol{M}$ is in $\boldsymbol{A}$ by (3-6), so if $\boldsymbol{M}$ has its inverse $\boldsymbol{M}^{-1}$, then $\boldsymbol{M}^{-1}$ must be also in $\boldsymbol{A}$. So we can write

$$\boldsymbol{M}^{-1}=\alpha\boldsymbol{I}+\beta\boldsymbol{J}+\gamma\boldsymbol{N}.$$

From (2-1) and (2-3) we have

$$\begin{aligned}\boldsymbol{M}\boldsymbol{M}^{-1} &= (d\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{N})(\alpha\boldsymbol{I}+\beta\boldsymbol{J}+\gamma\boldsymbol{N})\\ &= d\alpha\boldsymbol{I}+(\alpha+n\beta+d\gamma)\boldsymbol{J}+(-\alpha+d\gamma)\boldsymbol{N}-\gamma\boldsymbol{N}^2\\ &= d\alpha\boldsymbol{I}+(\alpha+n\beta+d\gamma)\boldsymbol{J}+(-\alpha+d\gamma)\boldsymbol{N}-\gamma[d\boldsymbol{I}+\lambda\boldsymbol{N}+\mu(\boldsymbol{J}-\boldsymbol{I}-\boldsymbol{N})]\\ &= (d\alpha-d\gamma+\mu\gamma)\boldsymbol{I}+(\alpha+n\beta+d\gamma-\mu\gamma)\boldsymbol{J}+(-\alpha+d\gamma+\mu\gamma-\lambda\gamma)\boldsymbol{N}\\ &= \boldsymbol{I}.\end{aligned}$$

So we have the following equations of α,β and γ,

$$d\alpha-(d-\mu)\gamma=1,\quad \alpha+n\beta+(d-\mu)\gamma=0,\quad -\alpha+(d+\mu-\lambda)\gamma=0$$

and solving these we have formula (3-7). □

From (3-7) we have the estimates $\widehat{w}_i$ of $w_i=\log w_i$ by the formula

$$\begin{bmatrix}\widehat{w}_1\\ \widehat{w}_2\\ \vdots\\ \widehat{w}_n\end{bmatrix}=\boldsymbol{M}^{-1}\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} \tag{3-8}$$

where $[y_1,y_2,\cdots,y_n]^t$ is the right hand side vector of the normal equation. By this formula we can calculate not only $\widehat{w}_i$ very simply, but also $\widehat{w}_i$ almost analytically $(i=1,\cdots,n)$. The formula (3-7) directly shows

$$V(\widehat{w}_i)/\sigma^2=M^{ii}=\alpha+\beta=\frac{n(d+\mu-\lambda)-2d+\lambda}{n[d(d+\mu-\lambda)+\mu-d]}=\frac{(n-2)d-(n-1)\lambda+n\mu}{n[d(d-1)-d\lambda+(d+1)\mu]}.$$

So we have the following theorem:

Theorem 2 *The variance of LLS estimate $\widehat{w}_i$ by SR design is given by*

$$V(\widehat{w}_i)=\frac{(n-2)d-(n-1)\lambda+n\mu}{n[d(d-1)-d\lambda+(d+1)\mu]}\sigma^2. \tag{3-9}$$

So this is independent of i $(i=1,\cdots,n)$.

Simplifying (3-9) by the formula (4-1) in the next section we have

$$V(\widehat{w}_i)=\frac{1}{n^2}\{\frac{n-1-d}{\mu}+\frac{(n-1)^2+d}{d}\}\sigma^2. \tag{3-10}$$

The precision of estimate $\widehat{w}_i$ is measured by its variances (3-10) or its standard deviation, the square root of (3-10).

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4. Quasi-SR Design

Through several calculations concernig the eigenvalues of the adjacent matrix $\boldsymbol{N}$ of an SR graph (n, d, λ, μ), we have the following relation [2],

$$d(d-\lambda-1) = (n-d-1)\mu \tag{4-1}$$

and the property such that the value of

$$\frac{1}{2}\left(n-1 \pm \frac{(n-1)(\mu-\lambda)-2d}{\sqrt{(\lambda-\mu)^2+4(d-\mu)}}\right) \tag{4-2}$$

must be integer [2].

So the value of the parameters n, d, λ and μ incur fairly strong constraints and can take only sparse values. So we generalize the concept of SR graph to r-th order regular graph whose adjacent matrix $\boldsymbol{N}$ with $\boldsymbol{I}$ and $\boldsymbol{J}$ generates r-th order (of $\boldsymbol{N}$) algebra, that is, $\boldsymbol{N}^{r+1}$ can be represented by the linear combination of $\boldsymbol{I}, \boldsymbol{N}, \cdots, \boldsymbol{N}^r$ and $\boldsymbol{J}$. First order regular graph is SR graph, and let us call r-th order regular graph as *quasi-SR* graph for $r \geq 2$. Of course any regular graphs are r-th order for some value of r. But when we say quasi-SR, the value of r must be smaller than n.

Example 2 The graph in (f) of Fig. 1 is a 2-nd order regular, and its adjacent matrix $\boldsymbol{N}_g$ has the following relation,

$$\boldsymbol{N}_g^3 = 3\boldsymbol{I} + \boldsymbol{N}_g - 3\boldsymbol{N}_g^2 + 6\boldsymbol{J}. \tag{4-3}$$

We do not ask the graph theoretic meaning of each coefficient in (4-3) any more. Of course generally if a graph is 2-nd order regular then its complement is also 2-nd order regular. For this case $\overline{\boldsymbol{N}_g} = \boldsymbol{J} - \boldsymbol{I} - \boldsymbol{N}_g$ has the relation,

$$\overline{\boldsymbol{N}_g}^3 = 4\overline{\boldsymbol{N}_g} + 6\boldsymbol{J}. \tag{4-4}$$

The value of $V(\widehat{\overline{w}}_i)$ on LLS for the design by $\overline{\boldsymbol{N}_g}$ is $V_g(\widehat{\overline{w}}_i) = 0.23438\sigma^2$ (independent of i).

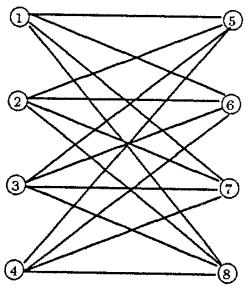

Fig. 3: The graph $\mathbf{G}(8, 4, 0, 4)$.

For the design by $\overline{\boldsymbol{N}_g}$, the number of objects $n = 8$ and the number of pairs compared $m = 16$. We have SR design with the same number $n = 8$ and $m = 16$, by the graph shown in Fig. 3 (which is called complete bipartite graph). This design gives the value of $V_{SR}(\widehat{\overline{w}}_i) = 0.21875\sigma^2$ (independent of i) by Theorem 2, which is smaller than $V_g(\widehat{\overline{w}}_i)$.

For quasi-SR design, information matrix for LLS always has the formula

$$\boldsymbol{M} = d\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{N},$$

so we can represent $\boldsymbol{M}^{-1}$ and $V(\widehat{\overline{w}}_i)$ as in like Theorem 1 and Theorem 2. But the formulas are rather complex, so if necessary we can calculate the numerical values of $\boldsymbol{M}^{-1}$ and $V(\widehat{\overline{w}}_i)$.

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5. Numerical Experiment (or Simulation)

We will show the goodness of SR or quasi-SR design by simulation. In order to do so, we compare the results of SR or quasi-SR designs with random design with the same value of n (the number of objects or points of the graph) and m (the number of pairs to be compared or edges of the graph), on two methods, LLS and the eigenvalue method (EV); where a random design with n and m is designed by a connected graph whose m edges are selected randomly.

As shown in Section 3, for LLS we can evaluate the goodness of a design by the variances of estimates of weights $w_1, \cdots, w_n$, but generally the value of $V(\widehat{w}_i)$ depends on i, so we take

$$V_{\max} = \max\{V(\widehat{w}_i) \mid i = 1, \cdots, n\} \tag{5-1}$$

where each $V(\widehat{w}_i)/\sigma^2$ can be calculated by the diagonal elements of $\boldsymbol{M}^{-1}$ the inverse of the information matrix $\boldsymbol{M}$. So we can evaluate the goodness based on only the design itself without considering data a_{ij}. Further the values of $V(\widehat{w}_i)$ for SR or quasi-SR designs of various parameters are shown in Table 7. Note that the value of $V(\widehat{w}_i)$ for SR or quasi-SR is independent of i, so $V_{\max} = V(\widehat{w}_i)$. So we have only to construct random designs with various values of n and m and calculate $V_{\max}$ by (5-1) and compare it with one of SR or quasi-SR designs. In our simulation we select the values of n and m shown in Table 3.

But on the EV method the goodness of the designs generally depends on the values of comparison data a_{ij}. The principle of our simulation to test the goodness of a design graph $\mathbf{G}$ is the following; For a given set of values of $w_1, \cdots, w_n$ satisfying $w_1 + \cdots + w_n = 1$, repeat the next process.

(a) Take a random number e_{ij} for each $(i, j) \in \mathbf{G}$ and calculate

$$a_{ij} = \frac{w_i}{w_j} e_{ij}, \quad a_{ji} = 1/a_{ij},$$

where $\log e_{ij}$ has normal distribution with $E[\log e_{ij}] = 0$, $V[\log e_{ij}] = \sigma^2$.

(b) Construct incomplete comparison matrix $\boldsymbol{A} = (a_{ij})$ with several missing elements, and calculte the maximal eigenvalue λ and an eigenvector $(\widehat{w}_1, \widehat{w}_2, \cdots, \widehat{w}_n)$ corresponding to λ, by Harker method or Two-stage method [3, 4, 5].

(c) Calculate $\overline{w}_i = \log w_i, \widehat{\overline{w}}_i = \log \widehat{w}_i$ $(i = 1, \cdots, n)$ and

$$\widehat{V}_{\max} = \max\{(\widehat{\overline{w}}_i - \overline{w}_i)^2 \mid i = 1, \cdots, n\}. \tag{5-2}$$

Repeat (a), (b), (c) many times (500 times in our simulation) and calculate the mean of $\widehat{V}_{\max}$ denoted by $\overline{\widehat{V}_{\max}}$. We take $\overline{\widehat{V}_{\max}}$ as the criterion of goodness of $\mathbf{G}$. The value of $(\widehat{\overline{w}}_i - \overline{w}_i)^2$ represents the square error of estimate $\widehat{\overline{w}}_i$ of $\overline{w}_i$ $(i = 1, \cdots, n)$, and the maximum of square errors $\max\{(\widehat{\overline{w}}_i - \overline{w}_i)^2 \mid i = 1, \cdots, n\}$ is the usual criterion of estimate. For example the minimax principle is to minimize the maximal error. So $\overline{\widehat{V}_{\max}}$ is a reasonable criterion.

Next problem is how to select the values of $w_1, \cdots, w_n$. T. L. Saaty [1] suggests that a_{ij} takes one of integers $1, 2, \cdots, 9$ and their inverses. So the maximal ratio among $w_1, \cdots, w_n$ must be 9. So it is typical pattern that $w_1, \cdots, w_n$ take values of $1, 2, \cdots, 9$. But in the simulation extrema cases are also important, and it is general tendency that the large value of n causes the wider range of values.

Considering these we select the values of $w_1, \cdots, w_n$ as follows. For $n = 10, 12, 15$, w_i takes one of $\{1, 2, 3, 4, 5\}$ in the narrow type and one of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in the wide type $(i = 1, \cdots, n)$. For $n = 18, 20$, w_i takes one of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in the narrow type and one of $\{1, 2, \cdots, 15\}$ in the wide type.

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Table 3: Results of simulation.

		Graph	LLS		HTM		TSM	
n	m		$\sqrt{V_{SR}}$	$\sqrt{V_r}$	$\sqrt{V_{SR}}$	$\sqrt{V_r}$	$\sqrt{V_{SR}}$	$\sqrt{V_r}$
10	15	$\overline{\mathbf{T}(5)}$	0.5831	1.0218	0.6373	0.8425	0.5532	0.7126
				1.0132	0.6375	0.8845	0.5223	0.5922
	25	$\mathbf{\Gamma}(2,5)$	0.4243	0.6002	0.4600	0.5319	0.4182	0.4710
				0.6091	0.4741	0.5549	0.4491	0.5201
	30	$\mathbf{T}(5)$	0.3905	0.4979	0.4326	0.4643	0.4029	0.4304
				0.5091	0.4386	0.4726	0.4129	0.4384
12	24	$\mathbf{Q}_1\mathbf{Q}_2$	0.5401	0.8575	0.7008	0.8185	0.5126	0.5936
				0.8524	0.6996	0.8036	0.5760	0.6661
	36	$\mathbf{\Gamma}(2,6)$	0.3909	0.5535	0.4976	0.5826	0.4561	0.5137
				0.5542	0.4921	0.5658	0.4541	0.5007
	54	$\mathbf{\Gamma}(4,3)$	0.3191	0.3711	0.3990	0.4034	0.3857	0.3885
				0.3722	0.4071	0.4216	0.3925	0.4014
	60	$\mathbf{\Gamma}(6,2)$	0.3028	0.3341	0.3774	0.3857	0.3706	0.3770
				0.3350	0.3742	0.3753	0.3677	0.3682
15	30	$\mathbf{Q}_1\mathbf{Q}_4$	0.5129	0.9268	0.6191	0.8444	0.5563	0.6550
				0.9325	0.6219	0.8610	0.6134	0.6922
	60	$\mathbf{T}(6)$	0.3464	0.4644	0.4290	0.4669	0.4092	0.4210
				0.4586	0.4346	0.4585	0.4238	0.4460
	75	$\mathbf{\Gamma}(3,5)$	0.3055	0.3726	0.3745	0.3972	0.3565	0.3783
				0.3736	0.3672	0.3912	0.3509	0.3674
	90	$\mathbf{\Gamma}(5,3)$	0.2789	0.3152	0.3419	0.3443	0.3336	0.3357
				0.3152	0.3423	0.3464	0.3343	0.3376
18	36	$\mathbf{Q}_1\mathbf{Q}_2$	0.6240	1.0131	1.2034	1.2166	0.7231	0.8342
				0.9966	1.2104	1.2598	0.9899	0.9956
	54	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.4422	0.6631	0.7957	0.8544	0.6525	0.6979
				0.6665	0.7900	0.8533	0.6135	0.6322
	81	$\mathbf{\Gamma}(2,9)$	0.3239	0.4427	0.5412	0.5902	0.4904	0.5198
				0.4414	0.5417	0.6134	0.4953	0.5735
	108	$\mathbf{\Gamma}(3,6)$	0.2805	0.3380	0.4722	0.4943	0.4438	0.4650
				0.3413	0.4754	0.4945	0.4467	0.4618
	135	$\mathbf{\Gamma}(6,3)$	0.2509	0.2785	0.4128	0.4220	0.4028	0.4091
				0.2790	0.4114	0.4147	0.4023	0.4031
	144	$\mathbf{\Gamma}(9,2)$	0.2430	0.2610	0.4101	0.4109	0.4054	0.4064
				0.2610	0.3999	0.4018	0.3947	0.3963
20	40	$\mathbf{Q}_1\mathbf{Q}_5$	0.5395	1.0278	0.7245	0.9768	0.5730	0.6853
				1.0334	0.7198	0.9998	0.7078	0.8221
	60	$\mathbf{Q}_1\mathbf{Q}_4\mathbf{Q}_8$	0.4292	0.6850	0.5879	0.6795	0.6401	0.6752
				0.6809	0.5761	0.6750	0.7758	0.7814
	80	$\mathbf{Q}_1\mathbf{Q}_3\mathbf{Q}_5\mathbf{Q}_7$	0.3484	0.5138	0.4622	0.5186	0.4272	0.4668
				0.5202	0.4556	0.5271	0.4449	0.5091
	100	$\mathbf{\Gamma}(2,10)$	0.3082	0.4141	0.4100	0.4472	0.3796	0.4137
				0.4113	0.4155	0.4519	0.3883	0.4527
	150	$\mathbf{\Gamma}(4,5)$	0.2517	0.2889	0.3315	0.3425	0.3230	0.3351
				0.2893	0.3361	0.3448	0.3253	0.3329
	160	$\mathbf{\Gamma}(5,4)$	0.2437	0.2743	0.3174	0.3253	0.3096	0.3172
				0.2736	0.3179	0.3215	0.3103	0.3132
	180	$\mathbf{\Gamma}(10,2)$	0.2297	0.2458	0.2966	0.2976	0.2941	0.2945
				0.2451	0.2963	0.2983	0.2941	0.2958

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The results of simulations are summarized in Table 3, where n is the number of objects (points of the graph) and m is the number of pairs compared (edges of the graph).

The symbols of graphs used for the designs, $\mathbf{T}(4), \mathbf{\Gamma}(2,5), \cdots$, etc. are explained in Section 6. The estimation methods are LLS, HM (Harker eigenvector method [3, 4]) and TSM (Two-stage eigenvector method [5]).

$\sqrt{V_{SR}}$: standard deviations of estimates of weights for SR or quasi-SR designs, calculated in Table 3 for LLS and calculated by (5-2) for eigenvector methods.

$\sqrt{V_r}$: standard deviations of estimates of weights for selected random designs, calculated by (5-1) for LLS and by (5-2) for eigenvector methods.

The upper part of each cell corresponds to the narrow type of weights and the lower part to the wide type.

The results show that SR or quasi-SR designs absolutely exceed random designs, that is without any exceptions that SR or quasi-SR designs give better results than random designs. Further the results show a specific feature as a byproduct; comparing HM with TSM, in almost cases TSM gives better results than HM, which is not main object of our research but is important to be mentioned.

6. Construction of SR and Quasi-SR Graphs

There are various methods to construct SR and quasi-SR graphs. Here we summarize them to cases (i) $\sim$ (vii); in cases (ii), (iii), (vi) and (vii) we construct SR graphs and these are mainly based on [2], in cases (i), (iv) and (v) we construct SR or quasi-SR graphs and these are based on our own idea.

(i) Cyclic graph:

Let $\{0, 1, \cdots, n-1\}$ be the set of points, then let us call a set of edges

$$\mathrm{C}_k = \{(i, i+k) \mid i = 0, 1, \cdots, n-1\}$$

a *cycle* with initial edge $(0, k)$, where $k = 1, 2, \cdots, (n-1)/2$ for odd n and $k = 1, 2, \cdots, n/2$ for even n, but $\mathrm{C}_{n/2} = \{(i, i+k) \mid i = 0, 1, \cdots, n/2-1\}$ for even n is called a *half cycle*. The set of edges of a complete graph is represented by

$$\begin{array}{ll} \mathrm{C}_1 \cup \mathrm{C}_2 \cup \cdots \cup \mathrm{C}_{(n-1)/2} & \text{for odd } n \quad \text{and} \\ \mathrm{C}_1 \cup \mathrm{C}_2 \cup \cdots \cup \mathrm{C}_{n/2} & \text{for even } n. \end{array}$$

Examples of $n = 6$ and $n = 7$ are shown in Table 4.

Table 4: Cyclic representation of edges.

	$n=6$					
C_1	(0,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,0)
C_2	(0,2)	(1,3)	(2,4)	(3,5)	(4,0)	(5,1)
C_3	(0,3)	(1,4)	(2,5)			

	$n=7$						
C_1	(0,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)	(6,0)
C_2	(0,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,0)	(6,1)
C_3	(0,3)	(1,4)	(2,5)	(3,6)	(4,0)	(5,1)	(6,2)

Let us call a graph (except a complete graph) composed of a set of cycles a *cyclic graph*. Thus for $n = 6$, $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_1 \cup \mathrm{C}_2, \mathrm{C}_1 \cup \mathrm{C}_3$, and $\mathrm{C}_2 \cup \mathrm{C}_3$ are all cyclic graphs. The adjacent matrix $\boldsymbol{N}$ of a cyclic graph is represented by the sum of basic cyclic matrices

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Table 5: Multiplication table.

$n=8$	$\boldsymbol{Q}_1$	$\boldsymbol{Q}_2$	$\boldsymbol{Q}_3$	$\boldsymbol{P}_4$
$\boldsymbol{Q}_1$	$\boldsymbol{Q}_2+2\boldsymbol{I}$	$\boldsymbol{Q}_1+\boldsymbol{Q}_3$	$\boldsymbol{Q}_2+2\boldsymbol{P}_4$	$\boldsymbol{Q}_3$
$\boldsymbol{Q}_2$		$2\boldsymbol{P}_4+2\boldsymbol{I}$	$\boldsymbol{Q}_1+\boldsymbol{Q}_3$	$\boldsymbol{Q}_2$
$\boldsymbol{Q}_3$			$\boldsymbol{Q}_2+2\boldsymbol{I}$	$\boldsymbol{Q}_1$
$\boldsymbol{P}_4$				$\boldsymbol{I}$

$(\boldsymbol{J}\boldsymbol{Q}_i=2\boldsymbol{J},\ \boldsymbol{J}\boldsymbol{P}_4=\boldsymbol{J})$

$n=9$	$\boldsymbol{Q}_1$	$\boldsymbol{Q}_2$	$\boldsymbol{Q}_3$	$\boldsymbol{Q}_4$
$\boldsymbol{Q}_1$	$\boldsymbol{Q}_2+2\boldsymbol{I}$	$\boldsymbol{Q}_1+\boldsymbol{Q}_3$	$\boldsymbol{Q}_2+\boldsymbol{Q}_4$	$\boldsymbol{Q}_3+\boldsymbol{Q}_4$
$\boldsymbol{Q}_2$		$\boldsymbol{Q}_4+2\boldsymbol{I}$	$\boldsymbol{Q}_1+\boldsymbol{Q}_4$	$\boldsymbol{Q}_2+\boldsymbol{Q}_3$
$\boldsymbol{Q}_3$			$\boldsymbol{Q}_3+2\boldsymbol{I}$	$\boldsymbol{Q}_1+\boldsymbol{Q}_2$
$\boldsymbol{Q}_4$				$\boldsymbol{Q}_1+2\boldsymbol{I}$

$(\boldsymbol{J}\boldsymbol{Q}_i=2\boldsymbol{J})$

$\boldsymbol{P}, \boldsymbol{P}^2, \cdots, \boldsymbol{P}^{n-1}$ $(\boldsymbol{P}^n=\boldsymbol{I})$, where $P_{i,j}$, (i,j) element of $\boldsymbol{P}$, is defined as; $P_{i,i+1}=1$ $(i=1,\cdots,n-1)$, $P_{n,1}=1$ and other elements are equal to 0. For example, adjacent matrices of C_1 and $\mathrm{C}_1 \cup \mathrm{C}_2$ are $\boldsymbol{P}+\boldsymbol{P}^{-1}$ and $\boldsymbol{P}+\boldsymbol{P}^{-1}+\boldsymbol{P}^2+\boldsymbol{P}^{-2}$ respectively. Let

$$\boldsymbol{Q}_i=\boldsymbol{P}^i+\boldsymbol{P}^{-i} \quad (i=1,\cdots,[n/2])$$

then $\boldsymbol{N}$ is always represented by sum of $\boldsymbol{Q}_i$s. And the results of adding and multiplying among $\boldsymbol{Q}_i$s are always represented by sum of $\boldsymbol{Q}_i$s and $\boldsymbol{I}$. The followings are multiplication tables of $\boldsymbol{Q}_i$s for $n=8$ and 9. (Note that for even n, we have $P^{n/2}=P^{-n/2}$, so $\boldsymbol{Q}_{n/2}=2\boldsymbol{P}^{n/2}$. Instead of $\boldsymbol{Q}_{n/2}$, we use $\boldsymbol{P}^{n/2}$ which is rewritten by $\boldsymbol{P}_{n/2}$.) Of course the table is symmetric so the under half is neglected).

Thus $\boldsymbol{N}, \boldsymbol{N}^2, \cdots$ are calculated by such multiplication tables, which is useful to get information about the albebra generated by $\boldsymbol{N}, \boldsymbol{N}^2, \cdots$. In Table 7, $\boldsymbol{Q}_i, \boldsymbol{Q}_i\boldsymbol{Q}_j$ and $\boldsymbol{Q}_i\boldsymbol{P}_k$ stand for the cyclic graphs $\mathrm{C}_i, \mathrm{C}_i \cup \mathrm{C}_j$ and $\mathrm{C}_i \cup \mathrm{C}_k$ respectively.

(ii) Triangular graph $\mathbf{T}(v)$:

Let V be a set of size v (≥ 3). Taking two elements a and b of V, and let the (unordered) pair $p=\{a,b\}$ be a point of the graph. Two points p and q are adjacent if and only if p and q have a common element. Triangular graph $\mathbf{T}(v)$ is always SR satisfying

$$n={}_vC_2, \quad d=2(v-2), \quad \lambda=v-2, \quad \mu=4,$$
$$m=nd/2=v(v-1)(v-2)/2 \quad \text{(the number of edges)}.$$

The complement $\overline{\mathbf{T}(v)}$ of $\mathbf{T}(v)$ is also SR, and its parameters are determined by (2-2).

(iii) Lattice graph $\mathbf{L}_2(v)$:

Let V be a set of size v (≥ 2). The set of points of a lattice graph is V^2 $(=V\times V)$, and two points $p=(a_1,a_2)$ and $q=(b_1,b_2)$ are adjacent if and only if p and q have a common coordinate $(a_1=b_1$ or $a_2=b_2)$. $\mathbf{L}_2(v)$ is always SR and satisfies

$$n=v^2, \quad d=2(v-1), \quad \lambda=v-2,$$
$$\mu=2, \quad m=nd/2=v^2(v-1).$$

The complement $\overline{\mathbf{L}_2(v)}$ is also SR and its parameters are determined by (2-2).

(iv) First kind k-dimensional lattice graph $\mathbf{L}_k(v)_l$ $(k\geq 3, v\geq 2, l=1,2,\cdots,k)$:

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Let V be a set of size v (≥ 2). The set of points is V^k, and two points $p = (a_1, a_2, \cdots, a_k)$ and $q = (b_1, b_2, \cdots, b_k)$ are adjacent if and only if Hamming distance of p and q is equal to l, i.e.,

$$|\{i \in \{i = 1, \cdots, k\} \mid a_i \neq b_i\}| = l.$$

$\mathbf{L}_k(v)_l$ is generally quasi-SR, where parameters are

$$n = v^k, \quad d = {}_kC_l(v-1)^l, \quad m = nd/2.$$

The complement $\overline{\mathbf{L}_k(v)_l}$ of $\mathbf{L}_k(v)_l$ is a quasi-SR graph of the same order as $\mathbf{L}_k(v)_l$.

(v) Second kind k-dimensional lattice graph $\mathbf{L}_k(v)_{\leq l}$ ($k \geq 3$, $v \geq 2$, $l = 2, \cdots, k-1$):
$\mathbf{L}_k(v)_{\leq l}$ is the same as $\mathbf{L}_k(v)_l$ except that the definition of the adjacency is "Hamming distance of two points is equal to or smaller than l". This is also quasi-SR, whose parameters are

$$n = v^k, \quad d = \sum_{i=1}^{l} {}_kC_i(v-1)^i, \quad m = nd/2.$$

The graph $\overline{\mathbf{L}_k(v)_{\leq l}}$ is isomorphic to $\mathbf{L}_k(v)_{\leq k-l}$.

(vi) Complete v-points k-partite graph $\mathbf{\Gamma}(k, v)$:
$\mathbf{\Gamma}(k, v)$ is the complement of disjoint union of k complete v-points graphs, that is, complete v-points k-partite graph. This is always a SR graph whose parameters are

$$n = kv, \quad d = (k-1)v, \quad \lambda = (k-2)v, \quad \mu = (k-1)v, \quad m = nd/2 = k(k-2)v^2/2.$$

An example for $k = 2, v = 4$ is shown in Fig. 3. The graph $\overline{\mathbf{\Gamma}(k, v)}$ is always separated graph.

(vii) Paley graph $\mathbf{P}(q)$:
Let q be a power of a prime number ($q = p^r$, where p is prime) and $q \equiv 1 \pmod 4$. Let $GF(q)$ be Galois field of size q (r-th order extension field of $GF(p)$). The points of the graph are elements of $GF(q)$, and two points a and b are adjacent if and only if $a - b$ is a square in $GF(q)$. This is always SR whose parameters are

$$n = q, \quad d = (q-1)/2, \quad \lambda = (q-5)/4, \quad \mu = (q-1)/4, \quad m = q(q-1)/4.$$

The graph $\overline{\mathbf{P}(q)}$ is isomorphic to $\mathbf{P}(q)$.

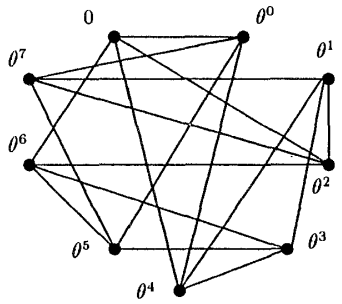

Fig. 4: The graph $\mathbf{P}(3^2)$.

Example of $\mathbf{P}(3^2)$ is shown in Fig. 4, where $GF(3^2)$ is the extension field of $GF(3) = \{0, 1, 2\} \bmod 3$, and θ is a primitive element of $\theta^2 = 1 + \theta$, and $\theta^0, \theta^2, \theta^4$ and θ^6 are squares in $GF(3^2)$.

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By the above mentioned methods we construct various SR or quasi-SR graphs and list up them in Table 7, where n is the number of points (the number with parenthesis is ${}_nC_2 = n(n-1)/2$), m is the number of edges, and d is the degree (or valency), and λ and μ are parameters for SR graph. The column denoted by "order" shows the order of the adjacent matrix $\boldsymbol{N}$. The symbols in the column "construction method" are explained in the above, but we abbreviate $\mathbf{Q}_i+\mathbf{Q}_j+\cdots+\mathbf{Q}_k$, $\mathbf{Q}_i+\mathbf{Q}_j+\cdots+\mathbf{P}_k$ as $\mathbf{Q}_i\mathbf{Q}_j\cdots\mathbf{Q}_k$, $\mathbf{Q}_i\mathbf{Q}_j\cdots\mathbf{P}_k$ respectively. The column denoted by "$\sqrt{V(w_i)}/\sigma$" shows the standard deviation of the estimate of weight for LLS, which is independent of data and only on the design.

The last column "$\sqrt{V(w_i)}/\sigma$ of complete" is the standard deviation of the estimate of weight for complete information case, which takes all of ${}_nC_2$ possible pairs. The adjacent matrix of complete design is $\boldsymbol{N} = \boldsymbol{J} - \boldsymbol{I}$ and $d = n-1$. By (3-6)

$$\boldsymbol{M} = d\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{N} = (n-1)\boldsymbol{I} + \boldsymbol{J} - (\boldsymbol{J} - \boldsymbol{I}) = n\boldsymbol{I}$$

and $\boldsymbol{M}^{-1} = 1/n\boldsymbol{I}$. So $\sqrt{V(w_i)}/\sigma = 1/\sqrt{n}$, where $V(w_i)$ is the abbreviation of $V(\widehat{w}_i)$.

In Table 8 we list up several adjacent matrices of the smaller n. This table is intended for practical usage for actual designs for AHP. But this has also important theoretical information. Considering that $\sqrt{V(w_i)}/\sigma$ is a criterion for goodness of designs we can say that SR is always the best design for the same values of n and m in the table, as conjectured in Section 1.

Further Table 6 shows that even an incomplete case, if the design is good, has sufficient precision comparable with complete information case. Here we pick up SR designs with about half value of m of complete design and compare the precision.

Table 6: Comparing precision.

n	8	10	12	14	16	18	20
ratio of $\sqrt{V(w_i)}$	0.354 / 0.467	0.316 / 0.424	0.289 / 0.391	0.267 / 0.364	0.250 / 0.342	0.236 / 0.324	0.229 / 0.315
ratio of m	28 / 16	45 / 25	66 / 36	91 / 49	120 / 64	153 / 81	171 / 95

Table 7: Some of SR graphs or quasi-SR graph.

n	m	d	λ	μ	order	construction method	$\sqrt{V(w_i)}/\sigma$	$\sqrt{V(w_i)}/\sigma$ of complete
5(10)	5	2	0	1	1	$\mathbf{P}(5) \cong \mathbf{Q}_1$	0.663325	0.447214
6	6	2			2	$\mathbf{Q}_1$	0.716860	0.408248
(15)	9	3	0	3	1	$\mathbf{\Gamma}(2,3) \cong \mathbf{Q}_1\mathbf{P}_3$	0.527046	
	12	4	2	4	1	$\mathbf{T}(4) \cong \mathbf{\Gamma}(3,2) \cong \overline{\mathbf{P}_3}$	0.456435	
7	7	2			2	$\mathbf{Q}_1$	0.769309	0.377964
(21)	14	4			2	$\mathbf{Q}_1\mathbf{Q}_2$	0.467130	
8	8	2			3	$\mathbf{Q}_1$	0.819680	0.353553
(28)	12	3			3	$\mathbf{Q}_1\mathbf{P}_4$	0.561010	
	12	3			2	$\mathbf{L}_3(2)_1$	0.563681	
	16	4			2	$\overline{\mathbf{L}_3(2)_1}$	0.484123	
	16	4			3	$\mathbf{Q}_1\mathbf{Q}_2$	0.481039	
	16	4	0	4	1	$\mathbf{\Gamma}(2,4) \cong \mathbf{Q}_1\mathbf{Q}_3$	0.467707	
	20	5			3	$\overline{\mathbf{Q}_1} = \mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_4$	0.420449	

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n	m	d	λ	μ	order	construction method	$\sqrt{V(w_i)}/\sigma$	$\sqrt{V(w_i)}/\sigma$ of complete
8	24	6	4	6	1	$\overline{\mathbf{P}_4} = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.381881	
9	9	2			3	$\mathbf{Q}_1$	0.867806	0.333333
(36)	18	4			3	$\mathbf{Q}_1\mathbf{Q}_2$	0.495440	
	18	4	1	2	1	$\mathbf{P}(9)$	0.484332	
	18	4			1	$\mathbf{L}_2(3)$	0.484322	
	27	6	3	6	1	$\mathbf{\Gamma}(3,3)$	0.384900	
	27	6			3	$\overline{\mathbf{Q}_4} = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.386097	
10	10	2			4	$\mathbf{Q}_1$	0.913783	0.3162277
(45)	15				4	$\mathbf{Q}_1\mathbf{P}_5$	0.594928	
	15	3	0	1	1	$\overline{\mathbf{T}(5)}$	0.583095	
	20	4			3	$\mathbf{Q}_1\mathbf{Q}_2$	0.510347	
	20	4			2	$\mathbf{Q}_1\mathbf{Q}_3$	0.485627	
	25	5			3	$\mathbf{Q}_3\mathbf{Q}_4\mathbf{P}_5 = \overline{\mathbf{Q}_1\mathbf{Q}_2}$	0.429052	
	25	5			2	$\mathbf{Q}_2\mathbf{Q}_4\mathbf{P}_5 = \overline{\mathbf{Q}_1\mathbf{Q}_3}$	0.444008	
	25	5	0	5	1	$\overline{\mathbf{\Gamma}(2,5)}$	0.424264	
	30	6			4	$\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4 = \overline{\mathbf{Q}_1\mathbf{P}_5}$	0.391101	
	30	6	3	4	1	$\mathbf{T}(5)$	0.390512	
	35	7			4	$\overline{\mathbf{Q}_1}$	0.359307	
	40	8	6	8	1	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4 = \overline{\mathbf{P}_5}$	0.335410	
	40	8	6	8	1	$\mathbf{\Gamma}(5,2)$	0.335410	
11	11	2			4	$\mathbf{Q}_1$	0.957787	0.301511
(55)	22	4			4	$\mathbf{Q}_1\mathbf{Q}_2$	0.525247	
	33	6			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3 = \overline{\mathbf{Q}_4\mathbf{Q}_5}$	0.397034	
	44	8			4	$\overline{\mathbf{Q}_1} = \mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.337580	
12	12	2			5	$\mathbf{Q}_1$	1.000000	0.288675
(66)	18	3			4	$\mathbf{Q}_1\mathbf{P}_6$	0.627878	
	24	4			3	$\mathbf{Q}_1\mathbf{Q}_2$	0.540062	
	24	4			5	$\mathbf{Q}_1\mathbf{Q}_3$	0.500481	
	30	5			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_6$	0.440242	
	36	6			3	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.403191	
	36	6	0	6	1	$\mathbf{\Gamma}(2,6)$	0.390868	
	42	7			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_6$	0.365190	
	48	8			4	$\overline{\mathbf{Q}_1\mathbf{P}_6} = \mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.339971	
	48	8	4	8	1	$\mathbf{\Gamma}(3,4)$	0.338502	
	54	9			5	$\overline{\mathbf{Q}_1} = \mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{P}_6$	0.319472	
	54	9	6	9	1	$\mathbf{\Gamma}(4,3)$	0.319192	
	60	10	8	10	1	$\overline{\mathbf{P}_6} = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.302765	
	60	10	8	10	1	$\mathbf{\Gamma}(6,2)$	0.302765	
13	13	2			5	$\mathbf{Q}_1$	1.040596	0.277350
(78)	26	4			5	$\mathbf{Q}_1\mathbf{Q}_2$	0.554677	
	39	6			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.409557	
	39	6	2	3	1	$\mathbf{P}(13)$	0.399704	
	52	8			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4 = \overline{\mathbf{Q}_5\mathbf{Q}_6}$	0.342721	
	65	10			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5 = \overline{\mathbf{Q}_6}$	0.304056	
14	14	2			6	$\mathbf{Q}_1$	1.079730	0.267261
(91)	21	3			6	$\mathbf{Q}_1\mathbf{P}_7$	0.659640	
	28	4			6	$\mathbf{Q}_1\mathbf{Q}_2$	0.569061	
	35	5			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_7$	0.449785	
	42	6			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.416072	
	49	7			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_7$	0.369799	

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n	m	d	λ	μ	order	construction method	$\sqrt{V(w_i)}/\sigma$	$\sqrt{V(w_i)}/\sigma$ of complete
14	49	7	0	7	1	$\mathbf{\Gamma}(2,7)$	0.364216	
	56	8			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.345930	
	63	9			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{P}_7$	0.323064	
	70	10			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.305432	
	77	11			3	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{P}_7$	0.290715	
	84	12	10	12	1	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.278174	
	84	12	10	12	1	$\mathbf{\Gamma}(7,2)$	0.278174	
15	15	2			6	$\mathbf{Q}_1$	1.117537	0.258199
(105)	30	4			5	$\mathbf{Q}_1\mathbf{Q}_2$	0.583189	
	45	6			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.422619	
	45	6	1	3	1	$\overline{\mathbf{T}(6)}$	0.401848	
	60	8			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.349206	
	60	8	4	4	1	$\mathbf{T}(6)$	0.346410	
	75	10			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.306951	
	75	10	5	10	1	$\mathbf{\Gamma}(3,5)$	0.305505	
	90	12			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.279016	
	90	12	9	12	1	$\mathbf{\Gamma}(5,3)$	0.278887	
16	16	2			7	$\mathbf{Q}_1$	1.154137	0.250000
(120)	24	3			7	$\mathbf{Q}_1\mathbf{P}_8$	0.690189	
	32	4			3	$\mathbf{L}_4(2)_1$	0.521666	
	32	4			7	$\mathbf{Q}_1\mathbf{Q}_2$	0.597054	
	32	4			7	$\mathbf{Q}_1\mathbf{Q}_4$	0.520076	
	40	5			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_8$	0.459302	
	48	6			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.429178	
	48	6	2	2	1	$\mathbf{L}_2(4)$	0.409840	
	56	7			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_8$	0.375467	
	64	8			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.352574	
	64	8	0	8	1	$\mathbf{\Gamma}(2,8)$	0.342327	
	72	9			2	$\overline{\mathbf{L}_4(2)_2}$	0.323554	
	72	9			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{P}_8$	0.326398	
	72	9	4	6	1	$\overline{\mathbf{L}_2(4)}$	0.324760	
	80	10			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.308666	
	88	11			3	$\underline{\mathbf{L}_4(2)_3}$	0.292961	
	88	11			3	$\overline{\mathbf{L}_4(2)_1}$	0.292961	
	88	11			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{P}_8$	0.292913	
	96	12			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.279897	
	96	12	8	12	1	$\mathbf{\Gamma}(4,4)$	0.279508	
	104	13			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{P}_8$	0.268642	
	112	14	12	14	1	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7$	0.258775	
	112	14	12	14	1	$\mathbf{\Gamma}(8,2)$	0.258775	
17	17	2			7	$\mathbf{Q}_1$	1.189632	0.242536
(136)	34	4			7	$\mathbf{Q}_1\mathbf{Q}_2$	0.610658	
	51	6			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.435723	
	68	8	3	4	1	$\mathbf{P}(17)$	0.348005	
	68	8			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.356031	
	85	10			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.310614	
	102	12			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.280842	
	119	14			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7$	0.259359	

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n	m	d	λ	μ	order	construction method	$\sqrt{V(w_i)}/\sigma$	$\sqrt{V(w_i)}/\sigma$ of complete
18	18	2			8	$\mathbf{Q}_1$	1.223582	0.235702
(153)	27	3			7	$\mathbf{Q}_1\mathbf{P}_9$	0.719566	
	36	4			7	$\mathbf{Q}_1\mathbf{Q}_2$	0.623954	
	45	5			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_9$	0.468172	
	54	6			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.442186	
	63	7			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_9$	0.380386	
	72	8			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.359513	
	81	9			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{P}_9$	0.329300	
	81	9	0	9	1	$\mathbf{\Gamma}(2,9)$	0.323942	
	90	10			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.312595	
	99	11			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{P}_9$	0.295304	
	108	12			6	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.281880	
	108	12	6	12	1	$\mathbf{\Gamma}(3,6)$	0.280542	
	117	13			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{P}_9$	0.270104	
	126	14			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7$	0.259964	
	126	14			4	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{P}_9$	0.250986	
	135	15	12	15	1	$\mathbf{\Gamma}(6,3)$	0.250924	
	144	16	14	16	1	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{Q}_8$	0.242956	
	144	16	14	16	1	$\mathbf{\Gamma}(9,2)$	0.242956	
19	19	2			8	$\mathbf{Q}_1$	1.257662	0.229416
(171)	38	4			8	$\mathbf{Q}_1\mathbf{Q}_2$	0.637088	
	57	6			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.448650	
	76	8			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.362541	
	95	10			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.314608	
	114	12			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.283033	
	133	14			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7$	0.260602	
	152	16			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{Q}_8$	0.243382	
20	20	2			9	$\mathbf{Q}_1$	1.290349	0.223607
(190)	30	3			8	$\mathbf{Q}_1\mathbf{P}_{10}$	0.747888	
	40	4			8	$\mathbf{Q}_1\mathbf{Q}_2$	0.649956	
	50	5			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_{10}$	0.477148	
	60	6			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$	0.455103	
	70	7			9	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{P}_{10}$	0.384918	
	80	8			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$	0.366639	
	90	9			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{P}_{10}$	0.332740	
	100	10			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5$	0.316700	
	100	10	0	10	1	$\mathbf{\Gamma}(2,10)$	0.308221	
	110	11			5	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{P}_{10}$	0.297482	
	120	12			9	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6$	0.284319	
	130	13			9	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{P}_{10}$	0.271759	
	140	14			7	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7$	0.261287	
	150	15			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{P}_{10}$	0.252017	
	150	15	10	15	1	$\mathbf{\Gamma}(4,5)$	0.251661	
	160	16			8	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{P}_9\mathbf{Q}_7\mathbf{Q}_8$	0.243819	
	160	16	12	16	1	$\mathbf{\Gamma}(5,4)$	0.243670	
	170	17			9	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{Q}_8\mathbf{P}_{10}$	0.236435	
	180	18	16	18	1	$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6\mathbf{Q}_7\mathbf{Q}_8\mathbf{Q}_9$	0.229734	
	180	18	16	18	1	$\mathbf{\Gamma}(10,2)$	0.229734	

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Table 8: Adjacent matrixes of some cyclic graph.

$C_1(n=5)=\mathbf{P}(5)$

$$\begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix}$$

$C_1(n=6)$

$$\begin{pmatrix} 0&1&0&0&0&1\\ 1&0&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 1&0&0&0&1&0 \end{pmatrix}$$

$C_1 \cup C_3(n=6)$

$$\begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}$$

$C_1 \cup C_2(n=6)$

$$\begin{pmatrix} 0&1&1&0&1&1\\ 1&0&1&1&0&1\\ 1&1&0&1&1&0\\ 0&1&1&0&1&1\\ 1&0&1&1&0&1\\ 1&1&0&1&1&0 \end{pmatrix}$$

$C_1(n=7)$

$$\begin{pmatrix} 0&1&0&0&0&0&1\\ 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 1&0&0&0&0&1&0 \end{pmatrix}$$

$C_1 \cup C_2(n=7)$

$$\begin{pmatrix} 0&1&1&0&0&1&1\\ 1&0&1&1&0&0&1\\ 1&1&0&1&1&0&0\\ 0&1&1&0&1&1&0\\ 0&0&1&1&0&1&1\\ 1&0&0&1&1&0&1\\ 1&1&0&0&1&1&0 \end{pmatrix}$$

$C_1(n=8)$

$$\begin{pmatrix} 0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&1&0 \end{pmatrix}$$

$C_1 \cup C_4(n=8)$

$$\begin{pmatrix} 0&1&0&0&1&0&0&1\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&1&0\\ 0&0&1&0&1&0&0&1\\ 1&0&0&1&0&1&0&0\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 1&0&0&1&0&0&1&0 \end{pmatrix}$$

$\mathbf{L}_3(2)_1$

$$\begin{pmatrix} 0&1&1&0&1&0&0&0\\ 1&0&0&1&0&1&0&0\\ 1&0&0&1&0&0&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&0&1\\ 0&0&0&1&0&1&1&0 \end{pmatrix}$$

$\overline{\mathbf{L}_3(2)_1}$

$$\begin{pmatrix} 0&0&0&1&0&1&1&1\\ 0&0&1&0&1&0&1&1\\ 0&1&0&0&1&1&0&1\\ 1&0&0&0&1&1&1&0\\ 0&1&1&1&0&0&0&1\\ 1&0&1&1&0&0&1&0\\ 1&1&0&1&0&1&0&0\\ 1&1&1&0&1&0&0&0 \end{pmatrix}$$

$\mathbf{\Gamma}(2,4)=C_1 \cup C_3(n=8)$

$$\begin{pmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{pmatrix}$$

$C_1 \cup C_2(n=8)$

$$\begin{pmatrix} 0&1&1&0&0&0&1&1\\ 1&0&1&1&0&0&0&1\\ 1&1&0&1&1&0&0&0\\ 0&1&1&0&1&1&0&0\\ 0&0&1&1&0&1&1&0\\ 0&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&1&0&0&0&1&1&0 \end{pmatrix}$$

$C_2 \cup C_3 \cup C_4(n=8)$

$$\begin{pmatrix} 0&0&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&1\\ 1&0&0&0&1&1&1&1\\ 1&1&0&0&0&1&1&1\\ 1&1&1&0&0&0&1&1\\ 1&1&1&1&0&0&0&1\\ 1&1&1&1&1&0&0&0\\ 0&1&1&1&1&1&0&0 \end{pmatrix}$$

$C_1 \cup C_2 \cup C_3(n=8)$

$$\begin{pmatrix} 0&1&1&1&0&1&1&1\\ 1&0&1&1&1&0&1&1\\ 1&1&0&1&1&1&0&1\\ 1&1&1&0&1&1&1&0\\ 0&1&1&1&0&1&1&1\\ 1&0&1&1&1&0&1&1\\ 1&1&0&1&1&1&0&1\\ 1&1&1&0&1&1&1&0 \end{pmatrix}$$

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7. Conclusion

(a) We proposed SR designs to select pairs in the whole set of ${}_nC_2$ pairs in AHP, and showed by simulation that SR design is the best of all other designs with the same value of n (the number of objects) and m (the number of pairs). (See Section 2, Table 3, Section 5.)

(b) An SR design corresponds to an SR graph. From the analytical properties of the adjacent matrix N of an SR graph, we give the theoretical formula to calculate the variance of estimates of weights for an SR design. (See Section 3.)

(c) The SR graph exists for only very sparse values of n and m. So we extend the SR graphs to the quasi-DR graphs which exist for wider values of n and m. It is shown by simulation that the quasi-SR design also gives good results. (See Section 4, Table 3, Section 5.)

(d) We surveyed the construction methods of SR graphs and further give new construction methods of SR and quasi-SR graphs (see Section 6). By these methods we construct SR and quasi-SR graphs and list up them with necessary parameters for $n = 5 \sim 20$ and various values of m. (See Table 7.)

(e) Table 7 gives not only practical designs for AHP, but also various theoretical information. This shows that SR or quasi-SR design with rather smaller values of m gives fairly good results compared with complete designs, and SR design is always better than quasi-SR with the same values of m and n. (See Table 7.)

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