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OPTIMAL SURVIVOR SEARCH FOR A TARGET WITH CONDITIONALLY DETERMINISTIC MOTION UNDER REWARD CRITERION

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Abstract In this paper, an optimal search-and-rescue operation maximizing the expected reward for a survivor with a conditionally deterministic motion and with a random lifetime is investigated. Necessary conditions for the optimal distribution of the searching effort and the optimal stopping time of the search are derived and the meaning of the conditions for the optimal plan are elucidated. To show the properties of the optimal search plan, several numerical examples are presented and special cases and generalization of the model are also discussed.

1. Introduction

In search-and-rescue operations (abbreviated as SAR) on the rough sea, it takes considerable time lag from an outbreak of shipwreck to starting of SAR operation usually and the survivor (called the target, hereafter) has been drifted away a long distance before the arrival of the searcher on the scene. Furthermore, considerable errors on the reported point of the distress and on the navigation of the searcher in the rough sea cannot be avoided. Hence, the search area may be expanded broadly and the life of target comes to a crisis. The searcher must find quickly the target in alive and a prompt and effective SAR operation is required. On the other hand, the searcher must prevent a duplicate distress on himself in the rough sea. Therefore, the search operation mentioned above is characterized by three factors: the moving target, the mortal target with a random lifetime and the prevention against a critical risk of the searcher. The survivor search problems have been studied by several authors. Stone [7], Nakai [5], Hohzaki and Iida [2] investigated optimal search for a stationary target with a random lifetime and Disenza & Stone [1] formulated the search for a mortal and moving target as a Markov chain defined on states of the target and proposed an algorithm to calculate the optimal distribution of searching effort maximizing the detection probability of the target. However, we cannot find any study dealing with the three factors stated above. In this paper, we consider the searcher's risk as his searching cost and formulate the search problem as a non-linear programming problem with the expected reward criterion. The expected reward is defined as the expected value gained by detection of the target being alive minus the expected searching cost. Here, the target value and the search cost are not necessary to be the cost expressed in a monetary unit. Any measure is acceptable if it can determine quantitatively both the importance of the successful SAR and the risk endured by the searcher. For example, if one accepts the view that the lives of the target and the searcher should be considered equally, then unit value is assigned to the target and the searching cost is defined by the probability of the searcher being lost at the rough sea per unit time. In any way, by setting the target value and the searching cost, we can define the expected reward as the measure of effectiveness and the model of SAR operation with the expected reward criterion is formulated.

2. Assumptions and Formulation of the model

2.1. Assumptions of the model

In order to formulate the model, we assume the following assumptions of the search process and define system parameters as follows.

(1). The target space consists of discrete cells: $\{i, i=1, \cdots, m\}$, and the time space consists of discrete time points: $\{t, t=0, 1, \cdots, N\}$, where N is the upper limit time of searching effort given to the searcher. The shipwreck is broken out at $t=0$ and the search is begun from $t=t_0$.

(2). It is assumed that there are finite number of possible paths on which the target is drifted. Let $i_k(t)$ be the cell of the target being at t on path k. (Cell $i_k(t)$ will be denoted by $\langle kt \rangle$ in subscript.) Then the path k is defined by a set of sells: $k=\{i_k(t), t=0, 1, \cdots\}$ and $K=\{k\}$. The target is drifted on path k with probability p_k, $\sum_k p_k=1$, and $\{p_k\}$ is assumed to be known by the searcher. (This rule of the target motion is a special case of the conditionally deterministic motion defined by paths.)

(3). A probability of the target being alive on the path k at the initial time is denoted by $Q_0(k)$ and a death probability of the target in cell i per unit time is given by $q(i)$. We assume the death of the target is independent in each time and cell.

(4). Let $\phi_i(t)$ be the searching effort allocated to cell i at time t, and the search plan is denoted by $\Phi=\{\phi_i(t), t=t_0, \cdots, T, i=1, \cdots, m\}$, where T is the stopping time of the search: $t_0 \leqq T \leqq N$.

(5). The conditional detection probability by $\phi_i(t)$ given the target being in cell i at time t is assumed to be given by the random search formula [6]: $1-\exp(-a_i\phi_i(t))$, where the detection rate a_i is assumed to be constant irrespective of the history of the past search and the state of the target being alive or not.

(6). The value of a live target detected at t is $R(t)$ (>0) and $R(t)$ is assumed to be a decreasing function of t. Here, we assume that the target value is zero if the detected target has died.

(7). The unit cost of searching effort in cell i is denoted by c_i. The total searching cost $\{C_0(t), t=t_0, \cdots, N\}$ is given to the searcher in advance of the search. We assume that $C_0(t)$ is continuously divisible in arbitrary when it is allocated among cells.

(8). The measure of effectiveness of search plan is assumed to be the expected reward which is defined as the expected value gained by detection of the live target minus the expected searching cost until stopping of the search. The search plan $\{\Phi, T\}$ is called optimum if it gives the maximum expected reward and is denoted with superscript *.

2.2. Formulation of the search problem

Let's consider a target drifting on path k. Since this target is in cell $i_k(t)$ at t and is searched by $\phi_{\langle kt \rangle}(t)$, the non-detection probability $\bar{P}(t,k)$ until t and the detection probability $p(t,k)$ at t are given as follows.

$$\bar{P}(t,k) = \exp(-\sum_{h=t_0}^{t} a_{<kh>}\phi_{<kh>}(h)), \quad (1)$$

$$p(t,k) = \{1-\exp(-a_{<kt>}\phi_{<kt>}(t))\}\exp(-\sum_{h=t_0}^{t-1} a_{<kh>}\phi_{<kh>}(h)).$$

The probability of this target being alive at t is given by

$$Q(t,k) = Q_0(k)\prod_{h=t_0}^{t-1}\{1-q(i_k(h))\}. \quad (2)$$

The searching cost $C(t,\Phi)$ used at t and the cumulative searching cost $D(t,\Phi)$ until t are presented as follows.

$$C(t,\Phi) = \sum_{i=1}^{m} c_i\phi_i(t), \quad (3)$$

$$D(t,\Phi) = \sum_{h=t_0}^{t}\sum_{i=1}^{m} c_i\phi_i(h).$$

Therefore, the conditional expected reward $G(\Phi,T|k)$ given the target path k is presented from Eqs. (1), (2) and (3) by

$$G(\Phi,T|k) = \sum_{t=t_0}^{T} p(t,k)\{Q(t,k)R(t)-D(t,\Phi)\} - \bar{P}(T,k)D(T,\Phi).$$

By calculating the weighted sum of $G(\Phi,T|k)$ by p_k, we have the expected reward $G(\Phi,T)$ as follows.

$$G(\Phi,T) = \sum_{k\in K} p_k[\sum_{t=t_0}^{T} p(t,k)\{Q(t,k)R(t)-D(t,\Phi)\} - \bar{P}(T,k)D(T,\Phi)]. \quad (4)$$

On the other hand, the search plan $\{\Phi,T\}$ stated above is restricted to the next constraints.

$$C(t,\Phi) \leqq C_0(t), \quad t = t_0,\cdots,N, \quad (5)$$

$$\phi_i(t) \geqq 0, \quad t = t_0,\cdots,T, \quad i = 1,\cdots,m,$$

$$t_0 \leqq T \leqq N.$$

Therefore, our problem is formulated as a non-linear programming problem to find the optimal search plan which maximizes the expected reward $G(\Phi,T)$ given by Eq. (4) subject to the restrictions (5).

3. Conditions for the optimal search plan

Necessary conditions of the optimal search plan will be derived by two steps. First, a conditionally optimal distribution of searching effort $\Phi_T^*=\{\phi_i^*(t|T)\}$ given the stopping time T of the search will be derived, and then, using Φ_T^* for each T, we will find the optimal stopping time T^* of the search.

3.1. The conditionally optimal distribution of searching effort

To simplify expressions, we define the following notations.

$$\bar{P}(t,k|T) = \exp(-\sum_{h=t_0}^{t} a_{<kh>}\phi_{<kh>}(h|T)), \quad (6\text{-}1)$$

$$\bar{P}_t(T,k) = \exp(-\sum_{h=t+1}^{T} a_{<kh>}\phi_{<kh>}(h|T)), \quad (6\text{-}2)$$

$$p_t(s,k) = \{1-\exp(-a_{<ks>}\phi_{<ks>}(s|T))\}\exp(-\sum_{h=t+1}^{s-1} a_{<kh>}\phi_{<kh>}(h|T)), \quad (6\text{-}3)$$

$$D_t(s,\Phi_T) = \sum_{h=t+1}^{s}\sum_{i=1}^{m} c_i\phi_i(h|T). \quad (6\text{-}4)$$

$$R_t(k,\Phi_T) = Q(t,k)R(t)+\sum_{h=t+1}^{T} p_t(h,k)\{D_t(h,\Phi_T)-Q(h,k)R(h)\}+\bar{P}_t(T,k)D_t(T,\Phi_T). \quad (6\text{-}5)$$

$\bar{P}(t,k|T)$ and $\bar{P}_t(T,k)$ are the non-detection probabilities of the target in $[t_0, t]$ and $[t+1, T]$, respectively, and $p_t(s,k)$ is the detection probability at s in the search $[t+1, s]$ when a search plan $\{\phi_i(t|T)\}$ is used. $D_t(s,\Phi_T)$ is the cumulative searching cost used in $[t+1, s]$. $R_t(k, \Phi_T)$ is the conditional expected gain given that the target selects path k and is detected at t. The first term in the r.h.s. of $R_t(k, \Phi_T)$ given by Eq.(6-5) is the expected value gained by the searcher when he detects the target at t, and the quantity given by summation of the second and the third terms is the expected risk of the search which is prearranged in $[t+1, T]$ if the target is not detected at t and this term is interpreted as the saved risk by the detection of the target at t. Therefore, $R_t(k, \Phi_T)$ is considered as the value which motivates the searcher to search in cell i at t.

Using terms defined by Eqs.(6)'s, the next theorem is presented.

[Theorem 1] *Necessary conditions of the conditionally optimal distribution of searching effort $\Phi_T^* = \{\phi_i^*(t|T)\}$ given the stopping time T ($\leqq N$) are that,*

if $\phi_i^*(t|T) > (=) 0$,

$$\frac{1}{c_i}\Big[\sum_{k\in B(t,i)} p_k\bar{P}(t-1,k)a_i\exp(-a_i\phi_i^*(t|T))R_t(k,\Phi_T^*) - c_i\sum_{k\in K} p_k\bar{P}(t-1,k)\Big]$$

$$= (\leqq)\ \mu(t), \quad \text{(the signs } (>,=) \text{ and } (=,\leqq) \text{ are same order.)} \tag{7}$$

$$\mu(t)\{C_0(t)-C(t,\Phi_T^*)\} = 0, \qquad C(t,\Phi_T^*) \leqq C_0(t), \qquad \mu(t) \geqq 0, \tag{8}$$

for all i and $t = t_0, \cdots, T$, where $B(t, i) = \{k \mid i_k(t)=i\}$: the set of paths passing i at t and $\mu(t)$ is a non-negative Lagrange multiplier determined by the constraint given by Eq.(5).

If $G(\Phi_T, T)$ is a concave function of Φ_T, the conditions mentioned above is the necessary and sufficient conditions for the optimal solution. □

(Proof) Necessary conditions (7) and (8) are derived by Lagrange's method of indeterminate coefficients. By setting slack variables and its multipliers corresponding to the inequality constraints (5), Lagrangean is defined, and then, Lagrangean is differentiated by $\phi_i(t|T)$, slack variables and its multipliers, and they are set to zero. Then, eliminating the slack variables, we have Eqs.(7) and (8). Here, if the objective function $G(\Phi_T, T)$ is a concave function of Φ_T, the conditions obtained above can be proved to be the necessary and sufficient conditions by direct application of the Kuhn-Tucker Theorem. (q.e.d.)

If the optimal search at t should not use the total searching cost exhaustively; $C_0(t) > C(t, \Phi_T)$ (referred to the partial search, hereafter), we have $\mu(t) = 0$ from Eq.(8). On the other hand, if $\mu(t) > 0$, $C_0(t) = C(t, \Phi_T^*)$ is derived, i.e., the optimal search should use all the total search cost (called the exhaustive search). By decomposing Eq.(8) by the exhaustive search and the partial search, Theorem 1 is rewritten as follows.

[Corollary 1-1] *Necessary conditions of the conditionally optimal distribution $\{\phi_i^*(t|T)\}$ being the exhaustive search at t are that Eq.(7) for all i and*

$$C(t, \Phi_T^*) = C_0(t), \tag{9}$$

and then, $\mu(t)$ is determined by Eq.(9).

For the optimal search being the partial search, if $\phi_i^*(t|T) > (=) 0$,

$$\sum_{k\in B(t,i)} p_k^{t-1}\bar{P}(t-1,k)a_i\exp(-a_i\phi_i^*(t|T))R_t(k,\Phi_T^*) = (\leqq)\ c_i, \tag{10}$$

where p_k^t is the posterior probability of the target path k at t given by

$$p_k^t = \frac{p_k\bar{P}(t,k)}{\sum_{k\in K} p_k\bar{P}(t,k)}. \tag{11}$$ □

(Proof) If the optimal search is exhaustive at t, Eq.(9) is obvious from the definition of the exhaustive search and Eq.(8) in Theorem 1 is interchanged by

Eq. (9) and $\mu(t)$ is determined by Eq. (9). If the optimal search is the pertial search, $\mu(t)=0$ from Eq. (8) and Eq. (10) is easily derived from Eq. (7) by setting $\mu(t)=0$ and applying Eq. (11). (q. e. d.)

3.2. The optimal stopping time of the search

Here, applying the conditionally optimal search plan $\Phi_T{}^*$ given by Theorem 1, we consider a search problem with a variable T and investigate an optimal stopping time T^0 which maximizes $G(\Phi_T{}^*, T)$ with respect to T subject to the constraints (5): $T^0 = \arg\max_T G(\Phi_T{}^*, T)$. Then, the optimal stopping time T^* is obtained by

$$T^* = \min\{T^0, N\}.$$

As for T^0, the next theorem is established.

[Theorem 2] *A necessary condition of T^0 maximizing $G(\Phi_T{}^*, T)$ is given that if $T \leqq (>) T^0$,*

$$\sum_{k\in K} p_k{}^{T-1}\{1-\exp(-a_{<kT>}\phi_{<kT>}{}^*(T|T))\}Q(T,k)R(T) \geqq (\leqq) \sum_{k\in K} p_k{}^{T-1}C(T, \phi_{<kT>}{}^*(T|T)), \qquad (12)$$

where the signs $(\leqq, >)$ in "if clause" and $(\geqq, \leqq)$ in Eq. (12) are applied in same order and $p_k{}^{T-1}$ is the posterior probability of the target path defined by Eq. (11). □

(Proof) Let's consider the case: $T \leqq T^0$. We set $T = T^0$ and $G(\Phi_T{}^*, T)$ is compared with $G(\Phi_{T-1}, T-1)$, where a search plan Φ_{T-1} is defined as a plan which truncates $\Phi_T{}^*$ at $T-1$. Then, we have the next relation.

$$G(\Phi_T{}^*, T) - G(\Phi_{T-1}, T-1)$$
$$= \sum_{k\in K} p_k\bar{P}(T-1,k)\{(1-\exp(-a_{<kT>}\phi_{<kT>}{}^*(T|T)))Q(T,k)R(T)-C(T,\phi_{<kT>}{}^*(T|T))\} \geqq 0.$$

The last inequality is deduced from the fact that T $(= T^0)$ is the optimal stopping time by the assumption stated above. From the above, we have the relation (12). Next, let's consider the case : $T > T^0$ and we define the conditional expected reward $G_t(\Phi_T, T, k)$ in $[t, T]$ given that the search Φ_T for the target selecting path k does not succeed in detection until t as

$$G_t(\Phi_T, T, k) = \sum_{h=t+1}^{T} p_t(h,k)\{Q(h,k)R(h)-D_t(h,\Phi_T)\} - \bar{P}_t(T,k)D_t(T,\Phi_T).$$

Here, setting $T = T^0 + 1$, we define Φ_T as a search plan such that $\overset{*}{\Phi}_{T^0}$ in $[t_0, T-1$ $(= T^0)]$ and $\phi_i{}^*(T^0|T^0)$ at $t = T$ $(= T^0+1)$, then we have the next relation.

$$G(\Phi_T{}^*, T) \geqq G(\Phi_{T-1}{}^*, T-1) + \sum_{k\in K} p_k\bar{P}(T-1,k)G_{T-1}(\phi_{<kT>}{}^*(T|T), T, k)$$
$$= G(\Phi_{T^0}{}^*, T^0) + \sum_{k\in K} p_k\bar{P}(T-1,k)[(1-\exp(-a_{<kT>}\phi_{<kT>}{}^*(T|T)))Q(T,k)R(T) -C(T,\phi_{<kT>}{}^*(T|T))].$$

From the above, we have

$$\sum_{k\in K} p_k\bar{P}(T-1,k)[(1-\exp(-a_{<kT>}\phi_{<kT>}{}^*(T|T)))Q(T,k)R(T)-C(T,\phi_{<kT>}{}^*(T|T))]$$
$$\leqq G(\Phi_T{}^*, T) - G(\Phi_{T^0}, T^0) \leqq 0.$$

The last inequality is deduced from the definition of the optimal stopping time T^0. From the above, we have Eq. (12). (q. e. d.)

We have obtained the conditionally optimal distribution of searching effort $\Phi_T{}^*$ and the optimal stopping time T^0 by Theorems 1 and 2, respectively. From these theorems, the optimal search plan (Φ^*, T^*) is determined by

$$(\Phi^*, T^*) = (\overset{*}{\Phi}_T{}^*, \min\{T^0, N\}).$$

3.3. An algorithm to calculate the optimal search plan

Unfortunately, we cannot solve analytically the simultaneous non-linear equations given by Eqs. (7) and (8), hence, we must calculate them numerically. As for the numerical solution of this type equations, a sequential approximation method called the Foreward-and-Backward algorithm (abbreviated as FAB algorithm) can be applied. This algorithm was presented by Washburn [9] to calculate the optimal distribution of searching effort maximizing the detection probability of a moving target with the constraint of total searching effort. In this problem, the constraint (5) for the total searching effort is satisfied in equality for the optimal solution. However, in our problem with the reward criterion, we must consider Eq. (8) since the constraint (5) holds sometimes for inequality. Hence, the FAB algorithm is modified by using Corollary 1-1 as follows.

In FAB algorithm, the searching effort $\{\phi_i(t|T), i=1,\cdots,m\}$ at t is calculated by assuming that the effort $\phi_i(\tau|T)$'s, $\tau\neq t$, except for $\phi_i(\tau|T)$'s, $\tau=t$, are all known (approximated by the last calculation in FAB). Then, if the exhaustive search is optimal, we have the next equation from Eq. (7).

If $\phi_i^*(t|T)>0$, $\quad K_{ti}\exp(-a_i\phi_i^*(t|T))-B_t=\mu(t)$, (13)

where $K_{ti}=\frac{a_i}{c_i}\sum_{k\in B(t,i)}p_k\bar{P}(t-1,k)R_t(k,\Phi_T^*)$ and $B_t=\sum_{k\in K}p_k\bar{P}(t-1,k)$.

Here, defining a notation $[x]^+$ by $[x]^+=x$, if $x>0$, and $[x]^+=0$, if $x\leqq 0$, we have $\phi_i(t|T)$ from Eq. (13) as

$$\phi_i(t|T)=\frac{1}{a_i}\left[\log\left(\frac{K_{ti}}{B_t+\mu(t)}\right)\right]^+. \tag{14}$$

In this case, since Eq. (9) holds, $\phi_i(t|T)$ given by Eq. (14) are substituted into Eq. (9) and we have

$$\sum_{i=1}^{m}\frac{c_i}{a_i}\left[\log\left(\frac{K_{ti}}{B_t+\mu(t)}\right)\right]^+=C_0(t). \tag{15}$$

$\mu(t)$ is solved numerically from the above equation and $\phi_i^*(t|T)$ is determined by Eq. (14).

On the other hand, if the partial search is optimal, then Eq. (10) holds and we have the next equation.

$$\phi_i(t|T)=\frac{1}{a_i}\left[\log\left(\frac{K_{ti}}{B_t}\right)\right]^+. \tag{16}$$

Since $\phi_i(t|T)$ at t is obtained by Eqs. (14) and (15) or (16), renewing $K_{t+1\,i}$, and B_{t+1} by $\phi_i(t|T)$, the calculation of FAB proceeds to the next time point $t+1$ (forward calculation). When $t=T$, $\{\phi_i(t|T), i=1,\cdots,m, t=t_0,\cdots,T\}$ is compared with $\{\phi_i(t|T)\}$ obtained by the previous iteration and if $\phi_i(t|T)$'s do not converge in a prescribed precision, the same calculation as stated above is repeated from $t=t_0$ to T (backward calculation) by renewing K_{ti} and B_t by the last $\phi_i(t|T)$. If $\{\phi_i(t|T)\}$ converges, Eq. (12) is checked. If $T\leqq T^0$, set $T=T+1$ and $\phi_i(t|T)$ is calculated by the FAB algorithm. If $T>T^0$, we set $T^*=T-1$ and $\phi_i^*(t|T^*)=\phi_i(t|T-1)$ and the FAB calculation is completed. The solution obtained by the FAB algorithm stated above is optimal if the objective function $G(\Phi_T,T)$ is a concave function as such cases discussed in §4. On the other hand, if $G(\Phi_T,T)$ is not concave as such a case that the target paths intersect with each other, the FAB algorithm gives only an exremal value.

4. The optimal search plan when the target paths have no intersection

In almost cases of the natural world, we may assume that the drifting paths of the target have no intersection (called separable). Then the next theorem is established from Theorem 1.

[Theorem 3] *If the possible paths of the target have no intersections, the objective function $G(\Phi_T, T)$ is a concave function of $\{\phi_i(t|T)\}$ and Theorem 1 gives the necessary and sufficient conditions of the conditionally optimal Φ_T^*, then the FAB algorithm stated before gives the optimal solution. In this case, Eqs. (7) and (8) are simplified slightly as follows.*

If $\phi_i^*(t|T) >(=) 0$ for $i = i_k(t)$,

$$\frac{1}{c_i}[\ p_k\bar{P}(t-1,k)a_i\exp(-a_i\phi_i^*(t|T))R_t(k,\Phi_T^*) - c_i\sum_{j=1}^{m} p_j\bar{P}(t-1,j)]$$
$$= (\leqq)\ \mu(t),\ \text{(the signs } (=,\leqq) \text{ are same order.)} \qquad (17)$$

$$\mu(t)\{C_0(t)-C(t,\Phi_T)\} = 0,\quad C(t,\Phi_T) \leqq C_0(t),\quad \mu(t) \geqq 0,\quad t = t_0,\cdots,T.\ \square$$

(Proof) If the possible paths of the target have not any intersections, the set of path $B(t,i) = \{k \mid i_k(t)=i\}$ has only one element at most, and therefore, differentiating twice $G(\Phi_T, T)$ with respect to $\phi_i(t)$, we have

$$\frac{\partial^2 G(\Phi_T,T)}{\partial\phi_i(t)^2} = -p_k\bar{P}(t-1,k)a_i^2\exp(-a_i\phi_i(t))R_t(k,\Phi_T) \equiv g(t,i),\quad i = i_k(t). \quad (18)$$

$$\frac{\partial^2 G(\Phi_T,T)}{\partial\phi_i(t)\partial\phi_j(t)} = 0,\quad i \neq j.$$

Therefore, Hessian of $G(\Phi_T, T)$ is obtained as follows.

$$H = \begin{bmatrix} g(t,1) & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & g(t,i) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & g(t,n) \end{bmatrix}.$$

Let a row vector ϕ be $(\phi_i(t), i=1,\cdots,n)$ and its transpose be ϕ^t. The concavity of $G(\Phi_T, T)$ with respect to ϕ is proved by showing $\phi H\phi^t \leqq 0$ for all ϕ $(\neq 0)$. From the above, we can easily calculate $\phi H\phi^t = \sum_i g(t,i)\phi_i^2(t)$. Here, we examine the sign of $g(t,i)$ defined by Eq. (18). Since $Q(h,k)R(h)$ is strictly decreasing, $R_t(k,\Phi_T)$ in Eq. (18) is proved to be positive as follows.

$$R_t(k,\Phi_T) = \{Q(t,k)R(t)+\sum_{h=t+1}^{T} p_t(h,k)(D_t(h,\Phi_T)-Q(h,k)R(h))+\bar{P}_t(T,k)D_t(T,\Phi_T)\}$$
$$> \{Q(t,k)R(t)(1-\sum_{h=t+1}^{T} p_t(h,k))+\sum_{h=t+1}^{T} p_t(h,k)D_t(h,\Phi_T)+\bar{P}_t(T,k)D_t(T,\Phi_T)\} > 0.$$

Hence, if there exists k, $i = i_k(t)$, then $g(t,i) < 0$, and otherwise, $g(t,i) = 0$ for any (t,i) from Eq. (18). Therefore, $\phi H\phi^t = \sum_i g(t,i)\phi_i^2(t) \leqq 0$ is proved and $G(\Phi_T, T)$ is a concave function of ϕ, and Eq. (17) gives the necessary and sufficient conditions for the conditionally optimal Φ_T^* by Theorem 1. (q. e. d.)

In many cases of local SAR operations, not only the target paths are separable but also the conditions of the environment may be uniform among cells. Then, the system parameters: $a_i, Q_0(k), q(i), c_i$, do not depend on cell and these are denoted by the same notations omitting the subscript. In this case, $Q(t,k)$ is denoted by $Q(t)$ since it depends only on time t and the search efficiency decreases as time elapse, because $Q(t)R(t)$ is a strictly decreasing function of t by the assumption (6), c and a are constant and the target paths have not any intersection. Therefore, the exhaustive search is optimal in the early stage of the search except the last time point (the partial search at the stopping time point is neglected hereafter). Here, we set the next constraints of the problem instead of Eq. (5).

$$
\begin{aligned}
&c\sum_{i=1}^{m} \phi_i(t) = C(t, \Phi) = C_0(t), \qquad t = t_0, \cdots, T-1,\\
&c\sum_{i=1}^{m} \phi_i(T) = C(T, \Phi) \leqq C_0(T),\\
&\phi_i(t) \geqq 0, \quad t = t_0, \cdots, T, \quad i = 1, \cdots, m,\\
&t_0 \leqq T \leqq N.
\end{aligned} \tag{19}
$$

In this case, the expected reward is presented as follows.

$$
G(\Phi, T) = \sum_{k\in K} p_k\Big[\sum_{t=t_0}^{T} p(t,k)\{Q(t)R(t) - \sum_{h=t_0}^{t} C_0(h)\} - \bar{P}(T,k)D(T,\Phi)\Big]. \tag{20}
$$

The next theorem is established to the problem maximizing $G(\Phi, T)$ given by Eq. (20) subject to Eq. (19).

[Theorem 4] *If the target paths have no intersection and the system parameters a, Q_0, q, and c do not depend on cell, the uniformly optimal search plan maximizing the detection probability always in the time interval such that $\sum_{h=t_0}^{t} C_0(h) \equiv D(t, \Phi) \leqq Q(t)R(t)$ is also the optimal search plan which maximizes the expected reward. The optimal stopping time T^* of the search is determined uniquely by $Q(T^*-1)R(T^*-1) > D(T^*-1, \Phi)$ and $Q(T^*)R(T^*) \leqq D(T^*, \Phi)$, if $Q(t_0)R(t_0) > D(t_0, \Phi)$. If $0 < Q(t_0)R(t_0) \leqq D(t_0, \Phi)$, the search should be stopped by the partial search at t_0 and if $Q(t_0)R(t_0) = 0$, the search should not be begun.* □

(Proof) Since the operator Σ's with respect to k and t in the first term of the r.h.s. of Eq. (20) can be exchanged each other, Eq. (20) is rewritten as follows.

$$
G(\Phi, T) = \sum_{t=t_0}^{T} [\{Q(t)R(t)-D(t,\Phi)\}\sum_{k\in K} p_k p(t,k)] - D(T,\Phi)\sum_{k\in K} p_k\bar{P}(T,k)]. \tag{21}
$$

The term in the r.h.s. of the above : $\{Q(t)R(t)-D(t, \Phi)\}$ is the reward when the target is detected at t and is a strictly decreasing function of t from $Q_0R(0) > 0$. The term : $\sum_k p_k p(t,k)$ is the detection probability of the target at t. Here, we consider the case in which the target paths do not intersect with each other and the environment is homogeneous. Then, we can define the target space by the paths $\{k\}$ instead of the cells $\{i\}$ and the moving target in $\{i\}$ is changed to the stationary target in $\{k\}$, and therefore, the search becomes the stationary search. On the other hand, in the stationary search, it has been proved that there exists a search plan called the uniformly optimal plan [6] in which the searching effort is distributed to maximize the detection probability $\sum_k p_k p(t,k)$ at each time point t in the time interval such that $\{Q(t)R(t)-D(t, \Phi)\} > 0$. This plan not only maximizes the first term in in the r.h.s. of Eq. (21) but also minimizes the second term: $D(T, \Phi)\sum_k p_k\bar{P}(T,k)$, since $\sum_k p_k\bar{P}(T,k)$ is the non-detection probability of the target in the search. Therefore, the uniformly optimal search plan maximizes the expected reward $G(\Phi, T)$. Hence, we have

$$
\max_\Phi G(\Phi, T) \rightleftarrows \max_\Phi \sum_k p_k p(t,k) \text{ at each } t, \text{ subject to Eqs. (19).}
$$

As for the optimal stopping time of the search, since $\{Q(t)R(t)-D(t, \Phi)\}$ is strictly decreasing, $G(\Phi, T)$ decreases if the search is continued after t such that $\{Q(t)R(t)-D(t, \Phi)\} < 0$ from Eq. (21). Therefore, the optimal stopping time is the time T^* determined uniquely by $Q(T^*-1)R(T^*-1) > D(T^*-1, \Phi)$ and $Q(T^*)R(T^*) \leqq D(T^*, \Phi)$ if $Q(t_0)R(t_0) > D(t_0, \Phi)$, and the search should be stopped at t_0 by the partial search if $D(t_0, \Phi) \geqq Q(t_0)R(t_0) > 0$, and the search should not be begun if $Q(t_0)R(t_0) = 0$. (q.e.d.)

5. Numerical Examples

In this section, several examples are analyzed to see the properties of the optimal search plan.

Suppose there exists four possible drifting paths of the target as shown in Fig. 1. Environment in cells for the survivor is severer in order (Cells 1 and 2), (Cell 3), (Cell 4) and the death rate $q(i)$ grows larger in this order. The total searching cost, the target value, the unit cost of searching effort and the parameter of the conditional detection probability are constant regardless of time point or cell and $Q_0(k)$ = 1 is assumed. We set a primary case: Case 1, and then, to see the sensitivity of the system parameters to the optimal search plan, other cases are analyzed by varying parameters of Case 1.

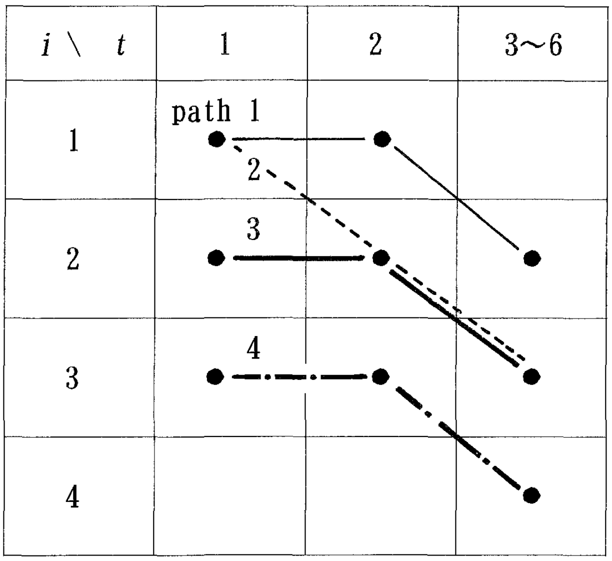

Fig. 1 Possible drifting paths of the target

(1). Case 1 (the primary case)

Probability distribution of target path: p_k = 0.25, k = 1, ⋯, 4,
Initial survive probability on path k: $Q_0(k)$ = 1, k = 1, ⋯, 4,
Distribution of death rate: $\{q(i)\}$ = {0.2, 0.2, 0.4, 0.6},
Starting time of the search: t_0 = 1,
Constraint of the total searching cost: $C_0(t)$ = 1, t = 1, 2, ⋯, 6, N = 6,
Unit cost of searching effort: c_i = 1, i = 1, ⋯, 4,
Detection rate: a_i = 1, i = 1, ⋯, 4,
The value of the live target: $R(t)$ = 7, t = 1, 2, ⋯, 6.

(2). Case 2 (the immortal target)

Death rate $q(i)$ is changed to $q(i)$ = 0 for all i and other parameters are same as Case 1.

Table 1 shows the optimal search plans for Cases 1 and 2. In Case 1, it should be noted that Path 4 is searched at the first time t = 1 only, and the optimal plan is the exhaustive search at t = 1 and the partial search at t = 2 and the optimal stopping time is T^* = 2. On the contrary to Case 1, in Case 2 (the search for the immortal target), the partial search is conducted at t = 1, 2, and thereafter, the exhaustive search is continued until the end of the search t = N. This may be explained by the fact that the posterior distribution of

Table 1 The optimal search plans for Cases 1 and 2

Case No.		1			2					
Changed parameter		Basic case			$q(i)=0,\ i=1,\cdots,4$ (Immortal target)					
	$i\backslash t$	1	2	3∼6	1	2	3	4	5	6
Φ^*	1	0.86	0	0	0.92	0	0	0	0	0
	2	0	0.85	0	0	0.65	0	0.26	0.33	0.33
	3	0.14	0	0	0	0.32	0.50	0.37	0.33	0.33
	4	0	0	0	0	0	0.50	0.37	0.33	0.33
$\mu(t)$		0.10	0	0	0	0	0.05	0.08	0.10	0.11
T^*		2			Stopped by the restriction $N=6$					
$G(\Phi_T^*, T^*)$		1.09			3.40					

the target is more equalized at $t=3,\cdots,6$ than that at $t=1,2$, and by the assumption of the exponential detection function. In this case, since the target does not die, the search effort is concentrated to the intersections of paths and is balanced, and after $t=5$, the effort is distributed uniformly since the posterior probability of the target becomes uniform. The expected reward is three times of Case 1. As shown in these cases, it should be noted that the optimal search plan and the expected reward are affected seriously by the lifetime of the target.

Next, we set Cases 3 ∼ 7 exchanging system parameters of Case 1 as follows.

(3). Case 3 : Sensitivity of the exponential detection rate
The exponential detection rate $a=1$ in Case 1 is changed to $a=2$.

(4). Case 4 : Sensitivity of the target value
The value of the live target $R=7$ in Case 1 is changed to $R=10$.

(5). Case 5 : Sensitivity of the unit searching cost
The unit cost of searching effort $c=1$ in Case 1 is changed to $c=2$.

(6). Case 6 : Sensitivity of the total searching cost
The total searching cost $C_0(t)=1$ in Case 1 is changed to $C_0(t)=2$.

(7). Case 7 : Sensitivity of the path distribution of the target
The uniform distribution of the target path $p_k=0.25$ for all k in Case 1 is changed to $\{p_k\}=\{0.1, 0.2, 0.3, 0.4\}$.

The optimal search plan of Cases 3 ∼ 7 are shown in Table 2. As shown in Case 3 of Table 2, the exponential detection rate a is very sensitive to both the optimal search plan and the expected reward. Since the target is detected easily compared with Case 1, the search becomes active and the total searching effort applied during the search is one and half times of that in Case 1 and the stopping time is prolonged to $T^*=4$. In Case 4, since the target value is large, the search is activated as similar to Case 3. However, in the same manner as Case 1, the target with a short lifetime on the path 4 is searched at the beginning of the search only. In Case 5, since the unit cost of the searching effort is expensive, the search becomes conservative. The total effort applied is diminished to one quarter of Case 1 and only the cells being the intersection of the target paths are searched. It is interesting that the result of Case 6 is almost the same as Case 1, i.e., the optimal search plan is not affected by the restriction of the total searching cost in this case. In Case 7, since the target selects the risky path : Path 4 with high probability, the search effort is concentrated to Path 4 at the beginning of the search, and then, Paths 2 and 3 are searched at t = 2 and the search is stopped. Path 1 is neglected since its probability is very small to search.

Table 2 The optimal search plans for Cases 3～7

Case No.		3					4			
Changed		a = 2					R = 10			
	i \ t	1	2	3	4	5, 6	1	2	3	4～6
Φ^*	i=1	0.59	0	0	0	0	0.79	0	0	0
	2	0.10	0.74	0.36	0.15	0	0	1.00	0.26	0
	3	0.31	0.26	0.24	0	0	0.21	0	0.13	0
	4	0	0	0	0	0	0	0	0	0
$\mu(t)$		0.25	0.04	0	0	0	0.49	0.16	0	0
T^*		4					3			
$G(\Phi_T^*, T^*)$		2.38					2.28			

Case No.		5			6			7		
Changed parameter		c = 2			$C_0(t)$ = 2			$\{p_k\}$={0.1, 0.2, 0.3, 0.4}		
	i \ t	1	2	3～6	1	2	3～6	1	2	3～6
Φ^*	i=1	0.33	0	0	0.95	0	0	0.28	0	0
	2	0	0.11	0	0	0.89	0	0.05	0.99	0
	3	0	0	0	0.24	0	0	0.67	0	0
	4	0	0	0	0	0	0	0	0	0
$\mu(t)$		0	0	0	0	0	0	0.07	0	0
T^*		2			2			2		
$G(\Phi_T^*, T^*)$		0.14			1.10			0.83		

6. Discussions

In this section, meanings of the conditions for the optimal search plan are elucidated and generalizations of the model and special cases are discussed.

6. 1. Interpretation of the conditions for the optimal search plan

1. The conditionally optimal distribution of searching effort

The conditions for the optimal effort distribution given by Theorem 1 are explained as follows.

Let's consider Eq. (8). This condition shows the complementary relation between $\mu(t)$ and the total searching cost $C_0(t)$. If the optimal search at t should not use all $C_0(t)$ (i. e., the partial search is optimal), $\mu(t)$ is zero. On the other hand, if $\mu(t) > 0$, the search should use all $C_0(t)$ at t (i. e., the exhaustive search is optimal). As stated later, $\mu(t)$ is a balance level of the ratio of the marginal expected reward vs. the unit cost when the exhaustive search is conducted. The meaning of $\mu(t)$ is elucidated as follows.

Suppose $\mu(t) > 0$ and the exhaustive search is optimal. Here, let's consider the meaning of the term in Eq. (7). The term in the l. h. s. of Eq. (7);

$$p_k\bar{P}(t-1,k)\,a_i\exp(-a_i\phi_i^*(t|T)) \tag{22}$$

is the increment of detection probability when the searcher adds unit searching effort to $\phi_i^*(t|T)$ in cell i at t given that the target selects path $k \in B(t, i)$ and is not detected until t. On the other hand, as mentioned before, the term $R_t(k, \Phi_T)$ in Eq. (7) given by Eq. (6-5) is the conditional expected gain given that the target path is $k \in B(t, i)$ and is not detected until that time. The summation with respect to $k \in B(t, i)$ of $R_t(k, \Phi_T)$ multiplied by Eq. (22) is the marginal expected gain at t. On the other hand, the term; $c_i \sum_k p_k\bar{P}(t-1, k)$ in Eq. (7) is the expected searching cost of unit effort used at t mentioned above, and the residual value obtained by the marginal expected gain minus the searching cost means the expected marginal reward at t. Hence, the l. h. s. of Eq. (7) is the ratio of the marginal expected reward versus the unit searching cost. Therefore, Eq. (7) is interpreted as that, if cell i is searched at t, the amount of the searching effort should be determined in such a way that the ratio of the marginal expected reward vs. the unit cost mentioned above is balanced to a level $\mu(t)$ (> 0) among the cells being searched at t, and if the searching effort should not be allocated to cell i at t, cell i does not have cost-effectiveness larger than $\mu(t)$.

Next, we consider the case in which the partial search is optimal ($\mu(t)=0$). In this case, Eq. (10) in Corollary 1-1 holds by substituting $\mu(t)=0$ into Eq. (7). As similar to $R_t(k, \Phi_T)$, the l. h. s. of Eq. (10) is the conditionally marginal expected gain given that the target is not detected until $t-1$, and the r. h. s. is the unit searching cost in cell i. Therefore, in the optimal partial search, the amount of the searching effort in each cell should be determined in such a way that the conditionally marginal expected gain is equilibrated to its unit cost and if the conditionally marginal expected gain is smaller than the unit cost, the cell should not be searched.

2. The optimal stopping time

Theorem 2 is elucidated as follows. Since p_k^{T-1} in Eq. (12) is the posterior probability of the path distribution, the l. h. s. of Eq. (12) is the conditional expected value gained by the searcher at the stopping time T given that the target is not detected until T-1 and the r.h.s. is the expected searching cost used at T. Therefore, this theorem states that if the stopping time T is before the optimal stopping time T^o, the conditional expected value gained by searcher at T is larger than the searching cost, and therefore, the searcher should not stop the search. On the other hand, if $T \geqq T^o$, the gain mentioned above is not larger than the searching cost, hence the search should not be continued.

6.2. Generalization of the model

1. Generalization of the spaces

In this paper, we assume the discrete target space and the discrete time space, and derive the theorems. However, these spaces can be generalized to the continuous target space and the continuous time space and similar theorems are also obtained. In this case, the problem is formulated in a variational problem and we can derive the conditions for optimal search plan by using Gateaux differential [8].

2. Generalization of the detection law

As mentioned in the assumption (5), we assume the exponential detection function: $f(\phi) = 1-\exp(-a_i\phi)$ which corresponds to the random search in each cell. However, this function is generalized to a more general function mentioned below and similar theorems are obtained without any difficulty [6].

$f(\phi)$: $f(0) = 0$, $f(\infty) = p \leqq 1$, continuous and differentiable with $f'(\phi) > 0$ and $f''(\phi) < 0$.

3. Discrete searching effort problem

In our model dealt with in this paper, the total searching cost $C_0(t)$ is assumed to be divisible arbitrarily. This implies that the searching effort is continuous such as search time, effective sweep rate, search area and so on. However, in the real world search problems, we can find many cases having the discrete effort. In this case, the problem becomes an integer programming problem including the combinatorial optimization problem. Although our theorems obtained in this paper cannot be applied directly to solve this problem, we may be able to utilize it to construct an efficient computational algorithm. Here, if we consider a relaxed problem with continuous variables for the integer programming problem, the theorems obtained here can be applied to solve the relaxed problem and its solution gives the upper bound estimation of the integer problem. Therefore, using these theorems, we can construct a branch-and-bound algorithm to compute the optimal integer solution.

6.3. Relation to the previous studies

1. If we set $Q_0(k) = 1$, $q(i) = 0$ in our model, we have $Q(t,k) = 1$ for all t and k. Then, the target becomes an immortal target and the theorems obtained in this paper are identical with the theorems of the optimal search for a moving target with the reward criterion studied by Iida & Hohzaki [3] and Iida [4] (discrete space version).
2. In our model, if we set $i_k(t) \equiv i_k(0)$ for all t and k, the target always stays at the initial position, and it meams the stationary target. In this case, our theorems stated before give the optimal solution of Nakai's model [5] (discrete space version).
3. Here, we consider a case with parameters: $R(t) = R \gg 1$, $c_i = 1$, and $i_k(t) = i_k(0)$ for any k and t. In this case, the search situation becomes a search for

a stationary target as mentioned above, and since R and c_i do not depend on cell by the assumption, the optimal distribution of searching effort maximizing the expected reward becomes identical with the uniformly optimal solution which maximizes always the detection probability during the search subject to the total search effort. Moreover, since $R \gg C(t, \Phi)$ by the assumption, $T^{o} \to \infty$ is concluded by Theorem 2, namely, the search is continued until $T^{*} = N$. Therefore, this case is identical with the Stone's model [7] which gives the optimal search plan maximizing the detection probability for a stationary mortal target (discrete space version).

7. Concluding Remarks

In this paper, we wants to throw light upon the optimal SAR operation which is characterized by three factors; moving of the target, lifetime of the target, and avoidance of the searcher's risk during the search. We formulate a search model for a target with the conditionally deterministic motion defined by paths and with a random lifetime under the expected reward criterion of the search. We derive necessary conditions for the optimal distribution of searching effort and the stopping time of the search, which give clear elucidations for the physical meaning. Numerical analysis of several examples give reasonable results expected intuitively. However, we are surprised by unexpected sensitivity of some system parameters on the optimal search plan. Here, it is important that we can evaluate the affect of the system parameters on the optimal solution quantitatively and explicitly by applying the theorems obtained here. We show that several results of previous studies are derived by specifying the system parameters of our model. In order to apply the theorems obtained in this paper to the real world SAR operations, we must clarify the definitions for the measures of the target value and the searching cost, and determine these values quantitatively from the actual data.

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