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MINIMAX LOCATION PROBLEM WITH A-DISTANCE

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Abstract In this paper a single facility location problem for an ambulance service station in a polygonal area X is considered. Our objective is to locate an ambulance service station so as to minimize the maximum distance of the route which passes from the facility to the hospital by way of the scene of accident. In this paper, we consider A-distance which is a generalization of rectilinear distance and was introduced by Widmayer *et al.*

Assuming m hospitals at the points $H_1, H_2, \cdots, H_m$ and denoting the nearest hospital to a point Q of $\boldsymbol{X}$ with $S(Q)$, the following problem $\mathbf{P}_{\mathrm{M}}$ is considered.

$$\mathbf{P}_{\mathrm{M}} : \underset{P^*}{\text{Minimize}} \max_{Q \in X} R(P^*, Q) = \{d_A(P^*, Q) + d_A(Q, S(Q))\},$$

where $P^* = (x^*, y^*)$ is the location of an ambulance service station to be determined. Then we show $\mathbf{P}_{\mathrm{M}}$ can be reduced to the messenger boy problem with A-distance. Utilizing this result, we propose an efficient solution procedure by extending Elzinga & Hearn Algorithm to A-distance case.

1 Introduction

In this paper a single facility location problem for an ambulance service station in a polygonal area $\boldsymbol{X}$ is considered. In modeling facility location problem, there are two main criteria; 1) minisum criterion 2) minimax criterion. In minisum criterion model the optimal location is determined so as to minimize weighted total distance to the demand points[5], [10]. In minimax criterion model the optimal location is determined so as to minimize the maximum distance between a new facility to be located and demand points. Emergency facilities location problem is often modeled as the minimax model. The mini-max models are studied by [4], [7], [12]. If an accident happens at a certain place, then ambulance servers rush to the scene of accident and take the injured persons to the hospital as soon as possible, then the objective is stated as the minimization of losses resulting from accidents. So we formulate our problem as the minimax location model.

In considering the location problem, the determination of measurement of the distance is also important. Two major distance measurements are used in many location studies. One is rectilinear distance in which the allowable orientations of travels are two orthogonal ones and this measurement is the most popular one in the urban area model. The other is Euclidean distance, with no restrictions of orientations to travel. The minimax location problems with Euclidean distance and rectilinear distance are already investigated by [3], [15], [2]. But neither measurement necessarily give the good approximation of distance in the urban travel distance cases. So in this paper, we consider A-distance case which is the class of block norm [14], [13], [11]. A-distance can be considered as a generalization of rectilinear distance and was introduced by Widmayer et al.[16]. They used A-distance in connection of VLSI design problem. They only investigated some properties of A-distance but did not apply it to any facility location problem.

In Section 2 definition of A-distance and some properties with respect to A-distance are given. In Section 3 we formulate our ambulance service station location problem and show that the problem is reduced to the messenger boy problem with A-distance. In Section 4 an efficient solution algorithm by extending Elzinga & Hearn Algorithm to A-distance case is presented. Finally we summarize this paper in Section 6.

2 A-distance

In facility location problem we measure the transportation cost by the distance between origin and destination. If we can travel to any orientations, good approximation of distance between two points may be Euclidean distance. But this ideal is seldom achieved in practice because of the existing of the some barriers to travel. When we measure the distance by rectilinear distance, the movement is allowed only to horizontal and vertical orientations. The Rectilinear distance is considered to be good approximation when distance is measured in city-street grid. The rectilinear distance $d_1(P_1, P_2)$ and Euclidean distance $d_2(P_1, P_2)$ between two points P_1 and P_2 are defined as follows.

$$d_1(P_1, P_2) = |a_1 - b_1| + |a_2 - b_2| \tag{2.1}$$
$$d_2(P_1, P_2) = ((a_1 - b_1)^2 + (a_2 - b_2)^2)^{1/2} \tag{2.2}$$

where $P_1 = (a_1, a_2)$, $P_2 = (b_1, b_2)$, a_1 and b_1 are x-coordinates of point P_1 and P_2, and a_2 and b_2 are y-coordinates of points P_1 and P_2 respectively.

In this paper we consider that travels are allowed only to some predetermined orientations $\boldsymbol{A}$, where $\boldsymbol{A}$ denote the set of allowable orientations. We measure distance between points by A-distance which is class of block norm and introduced by Widmayer et al.[16].

Let $\Theta(P_1, P_2)$ denote an orientation of a line, a halfline and a line segment which connect P_1 and P_2, if $\Theta(P_1, P_2)$ belongs to the set $\boldsymbol{A}$ then we call $\Theta(P_1, P_2)$ is A-oriented.

Then A-distance $d_A(P_1, P_2)$ between two points P_1 and P_2 can be defined as follows.

$$d_A(P_1, P_2) = \begin{cases} d_2(P_1, P_2) \ : \ \text{if } P_1 \text{ and } P_2 \text{ lie on an } A\text{-oriented line}, \\ \min_{P_3 \in \Re^2} \{d_A(P_1, P_3) + d_A(P_3, P_2)\} \ : \ \text{otherwise}. \end{cases} \tag{2.3}$$

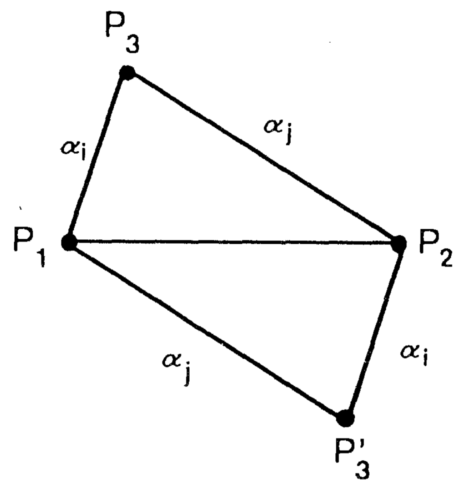

Figure 1. A-distance.

As is shown in Figure 1, $d_A(P_1, P_2)$ can be realized by a polygonal line segment consisting of at most two line segments by using at most one extra point, i.e., there exists a point P_3 such that

$$d_A(P_1, P_2) = d_2(P_1, P_3) + d_2(P_3, P_2). \tag{2.4}$$

Let $|\boldsymbol{A}| = r$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ denote the set of allowable orientations such that $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r < \pi$ where α_i, $i = 1, 2, \ldots, r$, represents the angle with the x-axis of the corresponding straight lines.

If $\alpha_i < \Theta(P_2, P_2) < \alpha_{i+1}$, where a_1 and b_1 are x-coordinates of point P_1 and P_2, and a_2 and b_2 are y-coordinates of points P_1 and P_2 respectively, then

$$d_A(P_1, P_2) = M_1 | m_2(a_1 - b_1) - (a_2 - b_2) | + M_2 | m_1(a_1 - b_1) - (a_2 - b_2) | , \tag{2.5}$$

where $m_1 = \max(\tan\alpha_i, \tan\alpha_{i+1})$, $m_2 = \min(\tan\alpha_i, \tan\alpha_{i+1})$, and

$$M_1 = \frac{\sqrt{1+m_1{}^2}}{m_1 - m_2} \quad , \qquad M_2 = \frac{\sqrt{1+m_2{}^2}}{m_1 - m_2} . \tag{2.6}$$

Note that either α_i or $\alpha_{i+1} = \pi/2$, then we interpret

$$M_1 = \lim_{m_1 \to \infty} \frac{\sqrt{1+m_1{}^2}}{m_1 - m_2} = 1 \quad , \qquad M_2 = \lim_{m_1 \to \infty} \frac{\sqrt{1+m_2{}^2}}{m_1 - m_2} = 0 \tag{2.7}$$

and

$$\lim_{m_1 \to \infty} M_2 m_1 = \lim_{m_1 \to \infty} \frac{m_1\sqrt{1+m_2{}^2}}{m_1 - m_2} = \sqrt{1+m_2{}^2} .$$

So, in this case,

$$d_A(P_1, P_2) = | m_2(a_1 - b_1) - (a_2 - b_2) | + \sqrt{1+m_2{}^2}\, | a_1 - b_1 | .$$

Thus, when $\alpha_i = 0$, $\alpha_{i+1} = \pi/2$, $d_A(P_1, P_2) = |a_2 - b_2| + |a_1 - b_1|$, since $m_2 = 0$, and it is very same as rectilinear distance.

For a point P, the locus of points P' with equal A-distance $\boldsymbol{d}$ from P is called the A-circle with radius $\boldsymbol{d}$ at center P, which has the boundary of the $2r$-gon, with corner points lying on the intersections of the circle with radius $\boldsymbol{d}$ at center P with the A-oriented lines through P, and edged between corner points adjacent on the circle(see Figure 2).

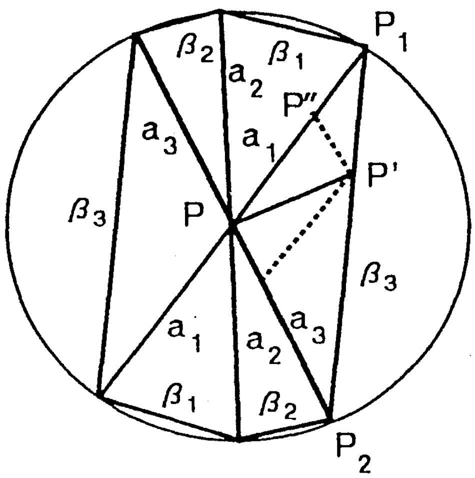

Figure 2. An A-circle with A-distance edges.

For two points P_1 and P_2, the bisector $B_A(P_1, P_2)$ of P_1 and P_2 is the locus of points that have equal A-distance to P_1 and P_2;

$$B_A(P_1, P_2) = \{P \in \Re^2 \mid d_A(P_1, P) = d_A(P_2, P)\} . \tag{2.8}$$

The center point of the line segment P_1P_2 belongs to the bisector $B_A(P_1, P_2)$. This point is called the anchor point of $B_A(P_1, P_2)$. The lines of all orientations of $\boldsymbol{A}$ through P_1 and through P_2 partition the plane $\Re^2$ into (bounded and unbounded) regions through which none of these lines passes, called fields. The bisector of two points P_1 and P_2, when restricted to a field of P_1 and P_2, is empty, a line segment, or a half line. The heavy polygonal line in figure 3 shows lower half of the bisector for points P_1 and P_2 and the star on that line is the anchor point.

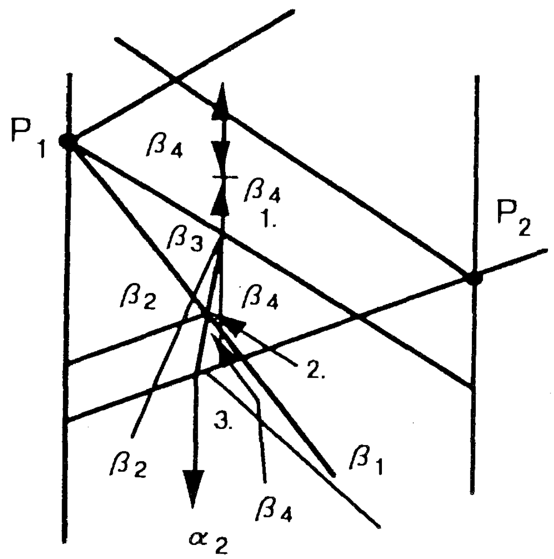

Figure 3. The march on the A-circle boundary when following the bisector.

Finally, $B_A(P_1, P_2)$ is constructed by an unbounded, continuous polygonal line, consisting of no more than $2r$-1 pieces. It partitions the plane $\Re^2$ into two unbounded regions, $BR_A(P_1 \mid P_2)$ and $BR_A(P_2 \mid P_1)$; where $P_1 \in BR_A(P_1 \mid P_2)$ and $P_2 \in BR_A(P_2 \mid P_1)$. All points in $BR_A(P_1 \mid P_2)$ are at least as close to P_1 as to P_2 and all points in $BR_A(P_2 \mid P_1)$ are at least as close to P_2 as to P_1.

For a set of v points, $\boldsymbol{P} = \{P_1, P_2, \ldots, P_v\}$, Voronoi polygon $V_A(P_i)$ on point P_i with respect to $\boldsymbol{P}$ with A-distance is defined as follows.

$$V_A(P_i) = \bigcap_{j \neq i} \{P \mid d_A(P, P_i) \leq d_A(P, P_j)\} . \tag{2.9}$$

The set of all Voronoi polygons for the points in $\boldsymbol{P}$ is a partition of the plane $\Re^2$ and is called the Voronoi diagram $VD_A(\boldsymbol{P})$ for $\boldsymbol{P}$. The boundary of $V_A(P_i)$ consists of partitions of bisector between P_i and P_j and is called Voronoi edge of $V_A(P_i)$. The endpoints of Voronoi edge are called Voronoi points. $VD_A(\boldsymbol{P})$ can be constructed in at most $O(v \log v)$

computational time. Figure 4 shows Voronoi diagram of case $\boldsymbol{A} = \{0, \pi/2\}$, i.e. rectilinear distance case.

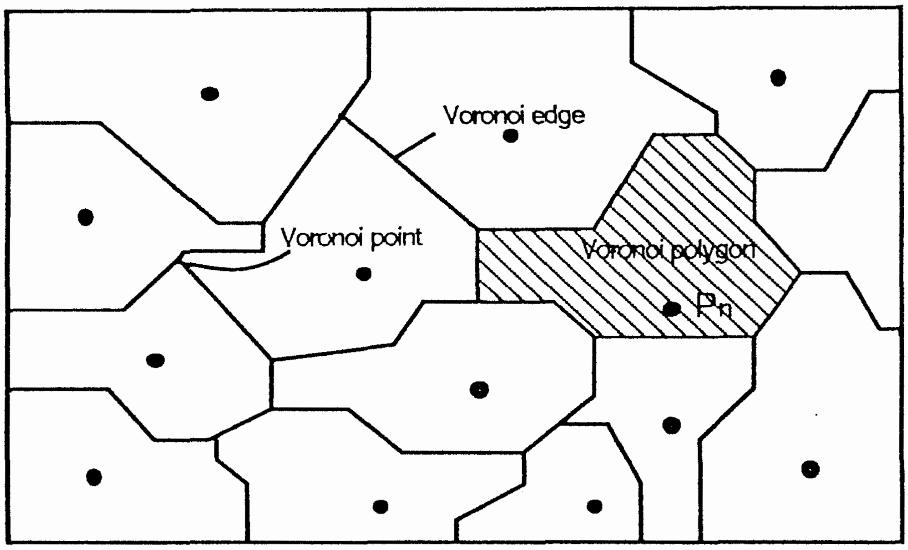

Figure 4. Voronoi Diagram $\boldsymbol{A}= \{0, \pi/2\}$.
(Rectilinear case)

3 Problem Formulation

Now we consider an ambulance service station location problem given as below.

If an accident (demand) happens, the ambulance servers rush to the scene of accident (demand point) and take the injured persons to the appropriate hospital as soon as possible.

We consider a polygonal area $\boldsymbol{X}$ in which an ambulance service station should be located, demand occur, and m hospital, $H_1, H_2, \ldots, H_m$, are existing. Since the place of demand point can not be predicted, we assume that demand points are distributed uniformly over the area $\boldsymbol{X}$.

Then our objective is to locate an ambulance service station so as to minimize the maximum A-distance of the route which passes from the service station to the hospital by way of the scene of accident.

Let $S(Q)$ denote the nearest hospital to the point Q, we formulate an ambulance service station location problem as following problem $\mathbf{P_M}$.

$$\mathbf{P_M} : \underset{P^*}{\text{Minimize}} \max_{Q \in \mathbf{X}} R(P^*, Q) = \{d_A(P^*, Q) + d_A(Q, S(Q))\}, \tag{3.1}$$

where $P^* = (x^*, y^*)$ is the location of an ambulance service station to be determined.

First, we construct Voronoi diagram $VD_{\mathrm{A}}(\boldsymbol{H})$ with respect to the set of points $\{\boldsymbol{H}\} = \{H_1, H_2, \ldots, H_m\}$ and A-distance in order to solve the problem. It can be done in at most $O(m \log m)$ computational time [16]. In the sequel, we show the candidate points which maximize $R(P^*, Q)$ are Voronoi points of Voronoi diagram $VD_{\mathrm{A}}(\boldsymbol{H})$ on the boundary of $\boldsymbol{X}$ or vertices of boundary of $\boldsymbol{X}$.

We call orientations α_i and α_{i+1} are adjacent and if $j = r$, we promise $j+1 = 1$. Then the following theorem holds.

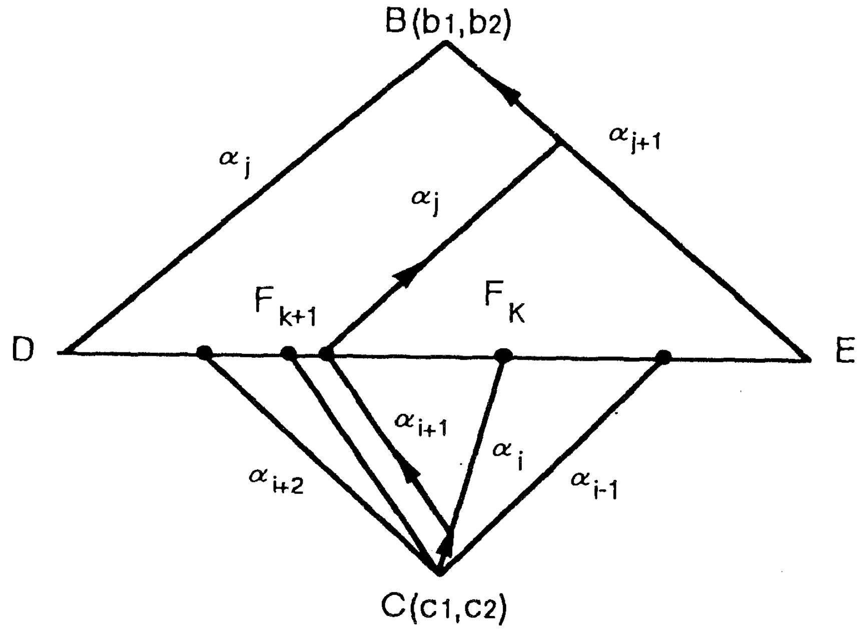

Figure 5. Relation of locations between line segment and some points.

Theorem 1. For the line segment DE with endpoints D, E and points B, C not on DE, suppose BD and BE are A-oriented adjacent orientations α_j, α_{j+1} (That is, consider the situation given in figure 5). Then the maximum A-distance among paths between B and C via point on the line segment DE is attained when the path visits D or E.

(Proof) The lines of all orientations of A through C partition the line segment DE into subintervals $[F_k, F_{k+1}]$, $k = 0, 1, \ldots, q-1$, where $F_0 = D$, $F_q = E$ and F_k, $k \neq 0, q$ are cross points between DE and all A-oriented lines through C. Consider the certain subinterval $[F_k, F_{k+1}]$. By a suitable transformation, we assume DE is x axis, $F_k = (0, 0)$, $F_{k+1} = (e, 0)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$ without any loss of generality. Then for point $T = (x, 0)$ $(0 \leq x \leq e)$

$$\begin{aligned} R_A^k(x) &= d_A(B, T) + d_A(T, C) \\ &= M_1|m_2(x - b_1) + b_2| + M_2|m_1(x - b_1) + b_2| \\ &\quad + M_3|m_4(x - c_1) + c_2| + M_4|m_3(x - c_1) + c_2|, \end{aligned} \tag{3.2}$$

where

$$m_1 = \max(\tan\alpha_j, \tan\alpha_{j+1}), \quad m_2 = \min(\tan\alpha_j, \tan\alpha_{j+1}), \tag{3.3}$$

$$M_1 = \frac{\sqrt{1 + {m_1}^2}}{m_1 - m_2}, \quad M_2 = \frac{\sqrt{1 + {m_2}^2}}{m_1 - m_2}. \tag{3.4}$$

$$m_3 = \max(\tan\alpha_i, \tan\alpha_{i+1}), \quad m_4 = \min(\tan\alpha_i, \tan\alpha_{i+1}), \tag{3.5}$$

$$M_1 = \frac{\sqrt{1+m_3{}^2}}{m_3 - m_4} \quad , \qquad M_2 = \frac{\sqrt{1+m_4{}^2}}{m_3 - m_4} \tag{3.6}$$

and α_j , α_{j+1} are the orientations corresponding to the subinterval $[F_k\,,\,F_{k+1}]$. $R_A^k(x)$ is a convex function and so maximum value of $R_A^k(x)$ is attained at $x = 0$ or $x = e$, i.e. $T = F_k$ or F_{k+1}. Thus the candidate points of maximum A-distance path is $F_0, \ldots, F_q$. Since each F_k, $k = 1, 2, \ldots, q-1$, $d_A(F_k\,,\,C) = d_2(F_k\,,\,C)$, $d_A(F_0\,,\,C) \geq d_2(F_0\,,\,C)$, $d_A(F_q\,,\,C) \geq d_2(F_q\,,\,C)$ and α_j , α_{j+1} are adjacent orientations, then $d_A(B\,,\,T) + d_2(T\,,\,C)$, for $T \in DE$ is consider as a path length between B and C via $T \in DE$.
Now let $D = (0\,,\,0)$, $E = (e'\,,\,0)$, $B = (b_1\,,\,b_2)$ and $C = (c_1\,,\,c_2)$ without any loss of generality. Then for $T = (x\,,\,0)$, $0 \leq x \leq e'$

$$\begin{aligned} d_A(B\,,\,T) + d_2(T\,,\,C) &= M_1|\,m_2(x-b_1) + b_2\,| + M_2|\,m_1(x-b_1)+b_2\,| \\ &\quad + \sqrt{(x-c_1)^2 + c_2{}^2}\ . \end{aligned} \tag{3.7}$$

Each term of right hand side in the above expression is convex function of x. So maximum of $d_A(B\,,\,T) + d_2(T\,,\,C)$ is attained at $x = 0$ or e', i.e. D or E. Further the path length through D or E is not less than $d_A(B\,,\,D) + d_2(D\,,\,C)$ or $d_A(B\,,\,E) + d_2(E\,,\,C)$, because $CD\,(EC)$ is not necessarily A-oriented. Therefore maximum is attained at D or E.

Q.E.D.

Further we relax the constraints that BD and BE have α_j and α_{j+1} orientations respectively from Theorem 1.

Theorem 2. Consider points B, C and line segment DE with endpoints D and E. Then $d_A(B\,,\,T) + d_A(T\,,\,C)$, $T \in DE$, is maximized when $T = D$ or E.

(Proof) We draw all A-oriented half lines from B and C, and let all intersections of these lines and DE be $T_1\,,\,T_2\,,\cdots,\,T_{t-1}$ by ordering from D. Further let $T_0 = D$ and $T_t = E$.

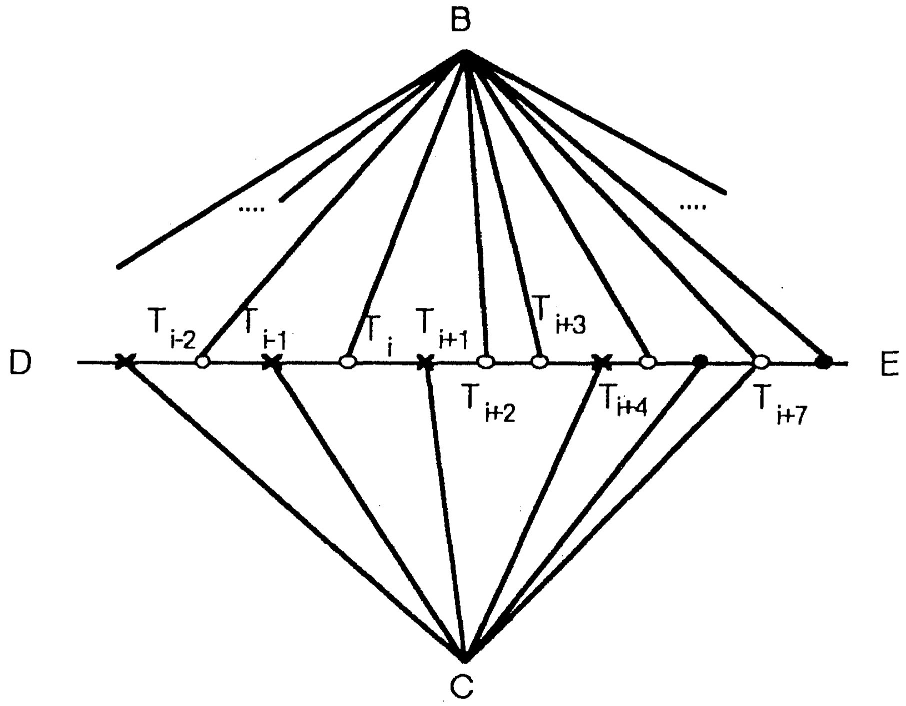

Figure 6. Intersections and line segment DE.

Then the situation may be as Figure 6. By Theorem 1, when consider the subinterval $T \in [T_{i-1}, T_{i+1}]$, $d_A(B, T) + d_A(T, C)$ is maximized at T_{i-1} or T_{i+1}. So T_i is dropped from candidates of maximizer. In turn, when considering $T \in [T_{i-2}, T_i]$, T_{i-1} is dropped by Theorem 1. Continuing this way, only remaining candidates are D, E and points as T_{i+7} which are intersections of DE and certain A-lines from both B and C. Let all points on DE with same property as T_{i+7} be $T_1', \ldots, T_l'$. Then

$$d_A(B, T_i') + d_A(T_i', C) = d_2(B, T_i') + d_2(T_i', C), \ i = 1, \ldots, l, \tag{3.8}$$

since both BT_i' and CT_i' are A-oriented. Since Euclidean distance is a convex function, then $d_2(B, T) + d_2(T, C)$, $T \in DE$ is maximized at $T = D$ or E. Thus

$$d_A(B, D) + d_A(D, C) \geq d_2(B, D) + d_2(D, C) \tag{3.9}$$

and

$$d_A(B, E) + d_A(E, C) \geq d_2(B, E) + d_2(E, C) \tag{3.10}$$

implies $d_A(B, T) + d_A(T, C)$, $T \in DE$ is maximized at D or E.

Q.E.D.

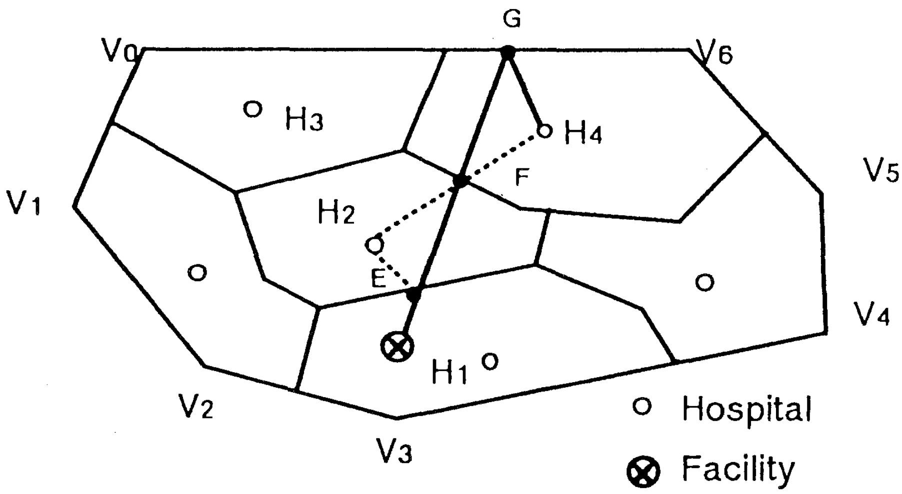

Figure 7. Voronoi diagram with respect to $\boldsymbol{H}$

Figure 7 illustrates a small example of Voronoi diagram with respect to $\boldsymbol{H} = \{H_1, \ldots, H_m\}$. Consider any interior point E of $\boldsymbol{X}$ on a Voronoi edge and draw the half line originating from the facility P and through E. Let the intersection of this half line and the other Voronoi edge of same Voronoi polygon as E be F.

Further let the intersection of this half line and boundary of $\boldsymbol{X}$ be G. It is sufficient to consider the situation of Figure 7, in order to show

$$d_A(P, G) + d_A(G, S(G)) \geq d_A(P, E) + d_A(E, S(E)). \tag{3.11}$$

It holds that

$$\begin{aligned} d_A(P, E) + d_A(E, S(E)) &\leq d_A(P, E) + d_A(E, F) + d_A(F, S(F)) \\ &= d_A(P, F) + d_A(F, S(F)), \end{aligned} \tag{3.12}$$

by the triangular property of A-distance.

Since F is on Voronoi edge of Voronoi polygons with respect to both H_2 and H_4, then

$$d_A(F\,,\,S(F)) = d_A(F\,,\,H_2) = d_A(F\,,\,H_4)\ . \tag{3.13}$$

Father

$$d_A(F\,,\,G) + d_A(G\,,\,S(G)) \geq d_A(F\,,\,H_4) \tag{3.14}$$

by the triangular inequality of A-distance. Thus

$$\begin{aligned} d_A(P\,,\,G) + d_A(G\,,\,S(G)) &= d_A(P\,,\,F) + d_A(F\,,\,G) + d_A(G\,,\,S(G)) \\ &\geq d_A(P\,,\,F) + d_A(F\,,\,H_4) \\ &= d_A(P\,,\,F) + d_A(F\,,\,S(F)) \\ &\geq d_A(P\,,\,E) + d_A(E\,,\,S(E))\ . \end{aligned} \tag{3.15}$$

From above consideration and Theorem 2, we have following Theorem 3.

Theorem 3. Candidates of maximizer of $R(P^*\,,\,Q)$ are

(a) Vertices of boundary of $\boldsymbol{X}$.

(b) The intersections of Voronoi edges and boundary of $\boldsymbol{X}$.

(Proof) It is directly shown from above consideration and Theorem 2.

Q.E.D.

From Theorem 3 we can reduce the number of demand points which shoud be considered in the solution procedure for $\mathbf{P_M}$ to finite size.

Let vertices of boundary of $\boldsymbol{X}$ be $V_1\,,\,V_2\,,\ldots,\,V_n$. Further let the intersections of Voronoi edges and boundary of $\boldsymbol{X}$ be $E_1\,,\,E_2\,,\ldots,\,E_e$. By a suitable numbering of $V_1\,,\,V_2\,,\ldots,\,V_n$ and $E_1\,,\,E_2\,,\ldots,\,E_e$, let those points be $Q_1\,,\ldots,\,Q_N$. N is the number of different points of them. Then by Theorem 3, $\mathbf{P_M}$ is reduced to the following messenger boy problem $\mathbf{P_E}$.

$$\mathbf{P_E}\ :\ \underset{P^*}{\text{Minimize}}\ \max_{Q_i\in\mathbf{X}}\{d_A(P^*\,,\,Q_i) + k_i \mid i = 1\,,\ldots,\,N\}\ , \tag{3.16}$$
$$\text{where}\quad P^* = (x^*\,,\,y^*)\quad\text{and}\quad k_i = d_A(Q_i\,,\,S(Q_i))\,,\ i = 1\,,\ldots,\,N\ .$$

Linear Programming type formulation of $\mathbf{P_E}$ is:

$$\underset{P^*}{\text{Minimize}}\ z\ , \tag{3.17}$$
$$\text{subject to}\quad d_A(P^*\,,\,Q_i) + k_i \leq z\,,\ i = 1\,,\ldots,\,N\ .$$

4 Solution Procedure for $\mathbf{P_M}$

In this section a solution procedure for $\mathbf{P_M}$ is given. As presented in the preceding section, $\mathbf{P_M}$ can be reduced to the equivalent location problem $\mathbf{P_E}$, then we give a solution procedure for $\mathbf{P_E}$ in order to solve $\mathbf{P_M}$.

Let C_i denote A-circle with radius k_i at center Q_i. Then $\mathbf{P_E}$ is further reduced to the determination of minimum radius A-circle covering all A-circles $C_1\,,\ldots,\,C_N$. In order to find this minimal covering A-circle, we define

$$C_A(C_i\,,\,C_j) \equiv \{p \in \Re^2 \mid d_A(Q_i\,,\,p) + k_i = d_A(p\,,\,Q_j) + k_j\}\ , \tag{4.1}$$
$$\text{for}\quad i \neq j\,,\ i\,,\ j = 1\,,\ldots,\,N\ .$$

$C_A(C_i, C_j)$ is a bisector between Q_i and Q_j taking account of the distance to hospitals. (See Figure 8)

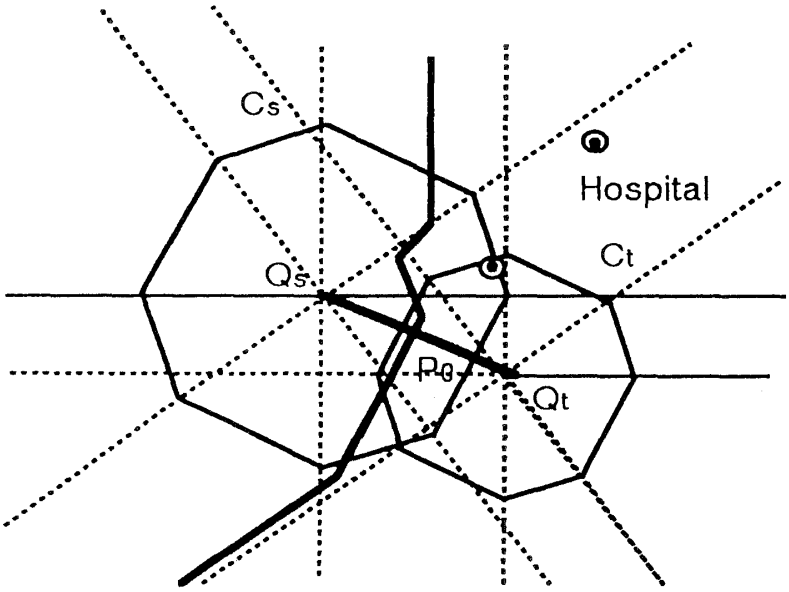

Figure 8. A bisector between Q_s and Q_t taking account of the distance to hospital

Then we have the following solution procedure.

Solution Procedure

Step 1: Draw A-circle $C_1, C_2, \ldots, C_N$ and let C_θ denote the biggest A-circle which has the largest radius among C_i, $i = 1, \ldots, N$. If C_θ covers all other C_i, $i \neq \theta$, then C_θ is the optimal A-circle. Stop(Q_θ is an optimal location of the ambulance service station (x^*, y^*)).
Otherwise, find C_s and C_t such that

$$\begin{aligned} & \max\{d_A(Q_i, Q_j) + k_i + k_j \mid i \neq j, i, j = 1, \ldots, N\} \\ = \; & d_A(Q_s, Q_t) + k_s + k_t \end{aligned} \tag{4.2}$$

and go to Step 2.

Step 2: Let P_0 be the intersection of $C_A(C_s, C_t)$ and the line segment connecting Q_s with Q_t. Draw the A-circle C_0 centered at P_0 with minimum radius covering C_s and C_t. If C_0 covers all C_i, then C_0 is an optimal A-circle. Stop (P_0 is an optimal location of the ambulance service station (x^*, y^*)).
Otherwise, choose one A-circle C_u which is not covered by C_0 and go to Step 3.

Step 3: Let P_0 be a intersection of $C_A(C_s, C_t)$, $C_A(C_t, C_u)$ and $C_A(C_u, C_s)$.
Draw A-circle C_0 covering C_s, C_t, C_u with minimum radius centered at P_0, that is externally tangent to these three A-circles.
If C_0 covers all C_i, then C_0 is an optimal A-circle. Stop(P_0 is an optimal location of the ambulance service station (x^*, y^*)).
Otherwise, choose one A-circle C_v which is not covered by C_0.

Step 4: Draw a half line from P_0 which through Q_v and let a intersection of the line and boundary be Z_v which is farthest from P_0. By same manner obtain Z_s, Z_t and Z_u. Let $D = Z_v$ and farthest point from D among Z_s, Z_t and Z_u be A. Divide a plane $\boldsymbol{X}$ into two half plane by line through both A and P_0. Let a point which does not

belongs to the same half plane with D be C.
Let $A = Q_s$, $C = Q_t$ and $D = Q_u$ and return to Step 3.

Theorem 4. If a set $\boldsymbol{A}$ is fixed, the above solution procedure finds an optimal location of a facility in at most $O(\max(n, m)^3 \cdot T)$ computational time, where T is the computational time constructing circum A-circle covering given three A-circles.

(Proof) By [16], a Voronoi diagram with A-distance for a set of m points in the plane can be constructed in $O(m \log m)$ time. The number of intersection points of Voronoi edges and boundary of $\boldsymbol{X}$ is $O(m)$ if a set $\boldsymbol{A}$ is fixed. Thus N is $O(\max(m, n))$. Validity of the solution procedure is clear from the above discussion, since certain three A-circles determine the optimal circum A-circles. In the worst case, $O(N^3)$ triplets of A-circles are tested for circum A-circles. Thus we have Theorem 4.

Q.E.D.

The following example illustrates the behavior of our solution procedure.

EXAMPLE
Consider the shade area $V_1V_2V_3V_4V_5$ and two hospitals in Figure 9.

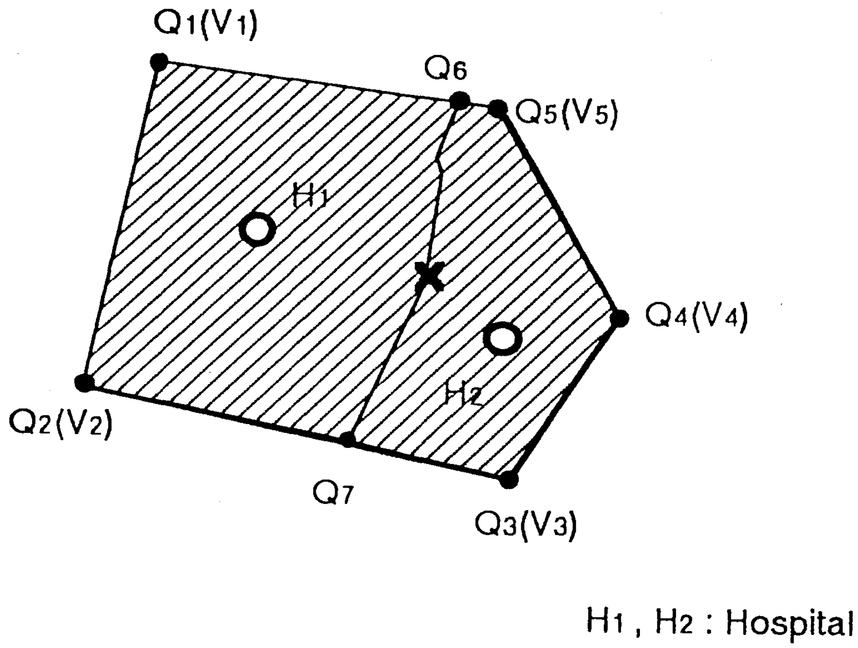

Figure 9. An example of the polygonal area $\boldsymbol{X}$

$\boldsymbol{A}$ is given in Figure 10. $N = 7(= 5 + 2)$.

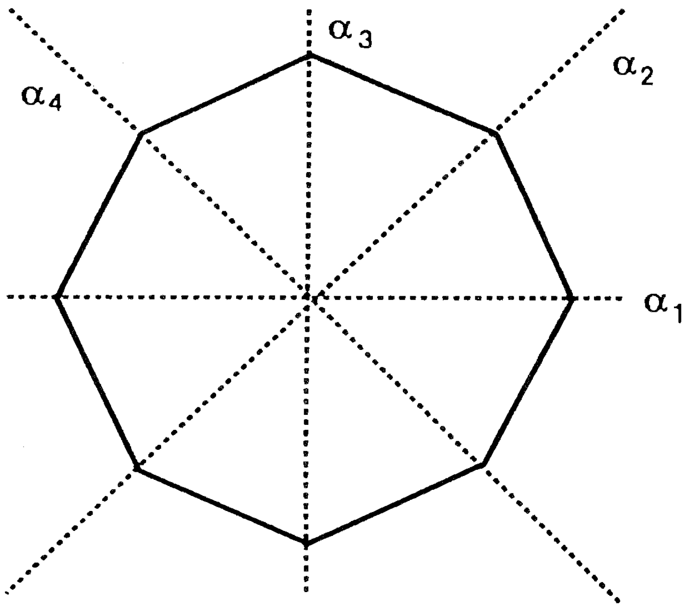

Figure 10. $\boldsymbol{A}=\{0\ ,\ 45\ ,\ 90\ ,\ 135\ \text{degree}\}$

Figure 11 illustrates $C_1, \ldots, C_7$ in Step 1.

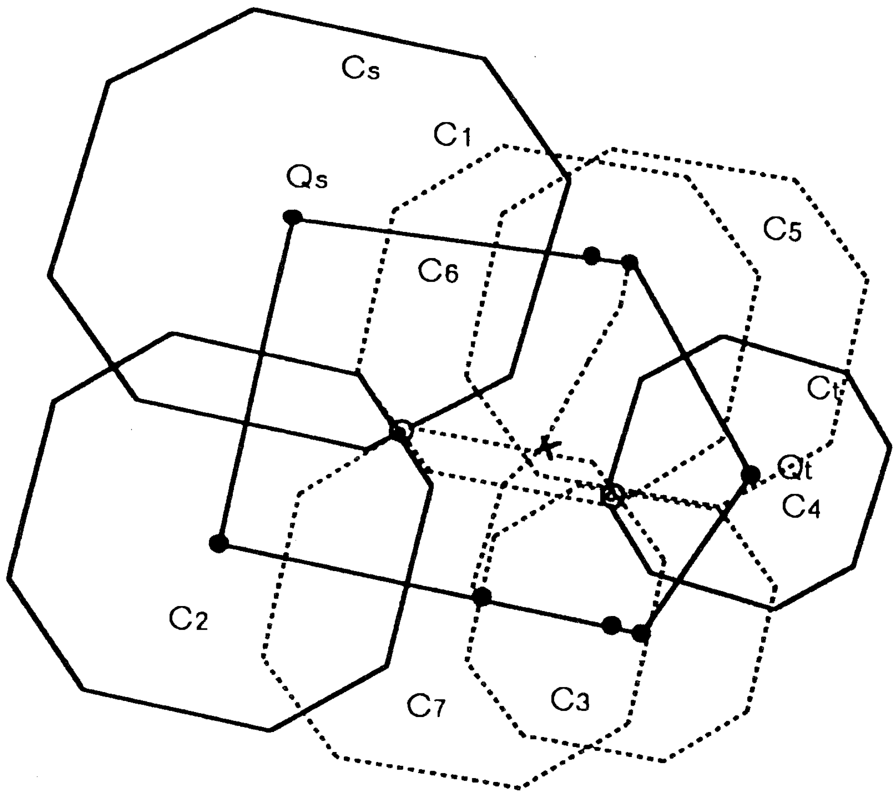

Figure 11. First iteration of Step 1.

$$\max\{d_A(Q_i, Q_j) + k_i + k_j \mid i \neq j\} = d_A(Q_1, Q_4) + k_1 + k_4 . \tag{4.3}$$

Thus we set $s = 1$, $t = 4$ and go to Step 2.
Figure 12 shows the result of Step 2, Step 3 and Step 4. C_2 and C_3 are not covered by C_0. We choose C_3 as C_u and go to Step 3. As a triplet of A-circle, we choose C_1, C_3 and C_4 and draw a circum A-circle in Step 3.

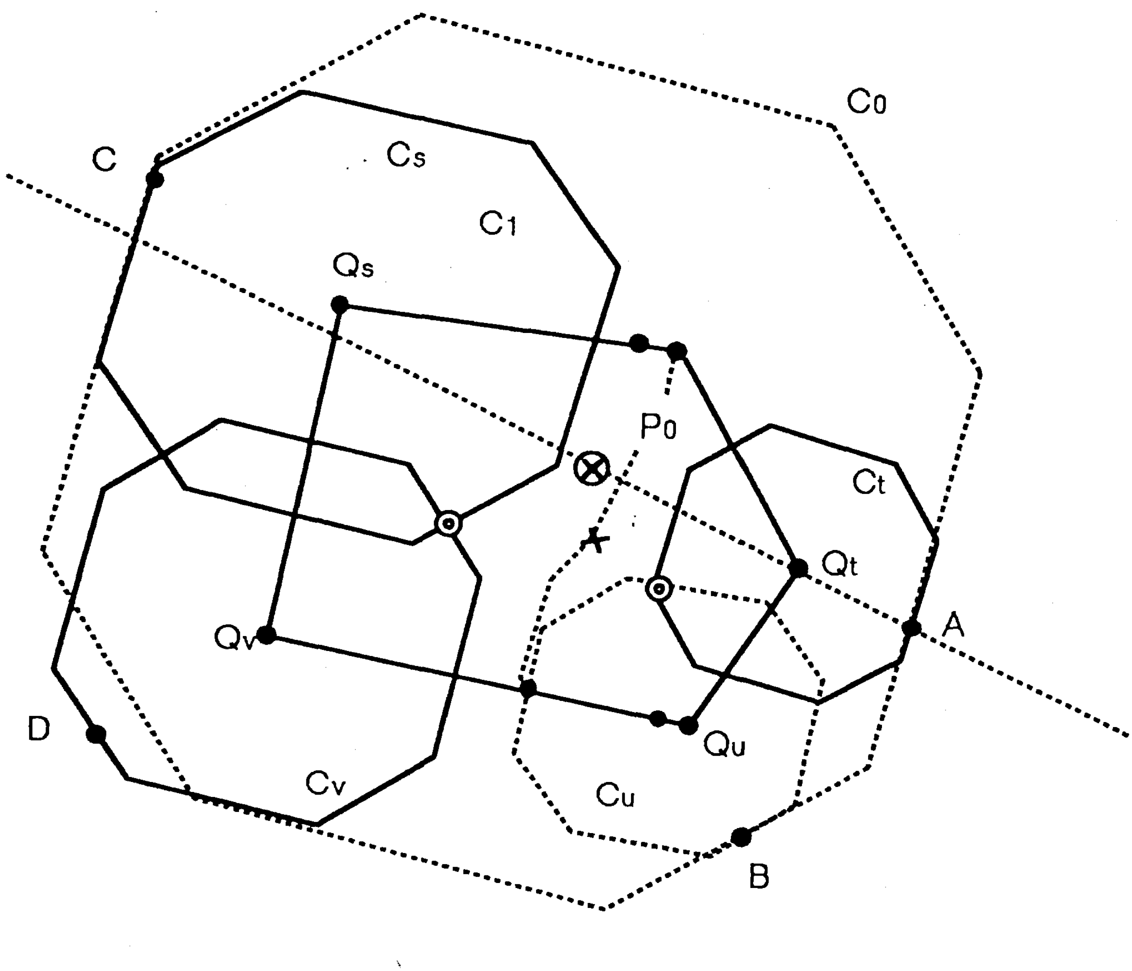

Figure 12. First iteration of Step 2, Step 3 and Step 4.

Figure 13 shows the result of Step 3. C_2 is not covered by C_0 again. So C_2 is chosen as C_v and go to Step 4. In Step 4, we choose $C_s = C_1$, $C_t = C_4$, $C_u = C_2$ and return to Step 3. In this iteration of Step 3, we obtain circum A-circle C_0 covering all C_i. Figure 13 shows this C_0 and P_0 is an optimal location (x^*, y^*) of the ambulance service station. Stop.

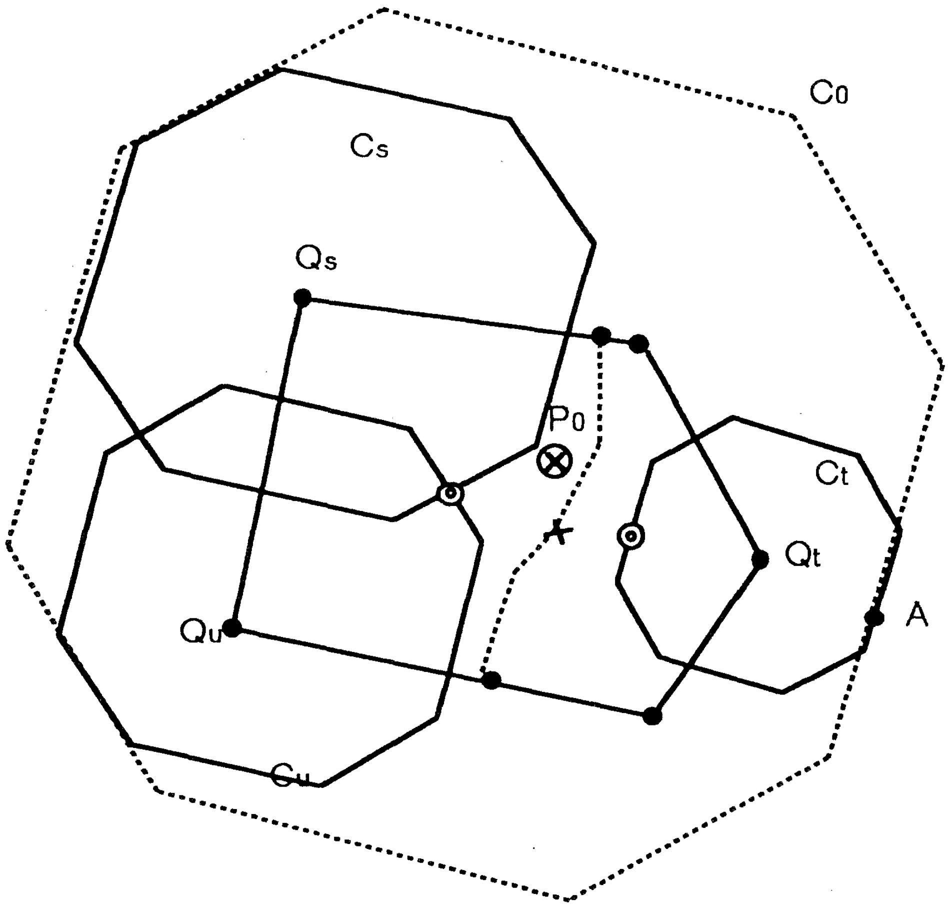

Figure 13. An optimal location of the facility.

5 Summary

We have considered a minimax facility location problem with A-distance. This is an extension of the rectilinear case discussed in [3]. But we could not find a suitable rule to choose a triplet of A-circles in Step 4 of our solution procedure. So, basically, we must check all triplets of A-circles for covering all C_i 's, in order to find an optimal location.

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