ON THE RELIABILITY OF GAVER'S PARALLEL SYSTEM SUSTAINED BY A COLD STANDBY UNIT AND ATTENDED BY TWO REPAIRMEN

Edmond J. Vanderperre
King Mongkut's Institute of Technology Ladkrabang

Abstract   We analyse the reliability of Gaver's parallel system sustained by a cold standby unit and attended by two identical repairmen. The system satisfies the usual conditions (i.i.d. random variables, perfect repair, instantaneous and perfect switch, queueing). Each operative unit has a constant failure rate but a general repair time distribution. Our reliability analysis is based on a time dependent version of the supplementary variable method.

We transform the basic equation into an integro-differential equation of the (mixed) Fredholm type. The equation generalizes Takàcs' integro-differential equation.

In order to present computational results, we outline the solution procedure for a repair time distribution with an *arbitrary* rational Laplace-Stieltjes transform. A particular numerical example displays the survivor function with the security interval that ensures a reliability level of at least 95%.

1.   Introduction

Two unit parallel systems (for instance, two power generators, in active redundancy [1], connected with the light-plant of a tunnel) are widely used to increase the reliability and safety of industrial plants. Gaver's two-unit parallel system [9] sustained by a cold or warm standby unit and attended by a *single* repair facility, henceforth called an S-system, has received considerable attention [4-8], [11-12].

As a variant, we analyse the reliability of Gaver's parallel system sustained by a cold standby unit and attended by *two* identical repairmen, henceforth called a T-system. The T-system satisfied the usual conditions (i.i.d. random variables, perfect repair [10], instantaneous and perfect switch [1], queueing).

Each operative unit has a constant failure rate but a general repair time distribution. Both repairmen are jointly busy, if and only if, at least two units are in failed state. In any other case, at least one repairman is idle.

It is evident that the T-system reduces the waiting time for repair with respect to a similar S-system. Therefore, a T-system improves the reliability of the corresponding S-system.

Our reliability analysis is based on a time dependent version of the supplementary variable method. We transform the basic equation into an integro-differential equation of the (mixed) Fredholm type. The equation generalizes Takàcs' integro-differential equation, e.g. [2].

In order to present computational results, we outline the solution procedure for a repair time distribution with an *arbitrary* rational Laplace-Stieltjes transform. A particular numerical example displays the survivor function with the security interval I that ensures a reliability level of at least 95%.

2. Formulation

Consider a T-system satisfying the usual conditions.

Each operative unit has a constant failure rate $\lambda > 0$ but a general repair time distribution $R(\bullet), R(0) = 0$. Let $R^-(\bullet) \equiv 1 - R(\bullet)$. Without loss of generality, (see forthcoming remark) we may assume that $R(\bullet)$ has a bounded density function defined on $[0, \infty)$.

Let $\{N_t, t \geq 0\}$ be a stochastic process with arbitrary discrete state space $\{A, B, C, D\} \subset [0, \infty)$ characterized by the following events:

$\{N_t = A\}$: "Both repairmen are idle at time t, i.e., two units are operating in parallel sustained by a cold standby unit." Fig. 1 shows a functional block-diagram of the T-system operating in the renewal state A.
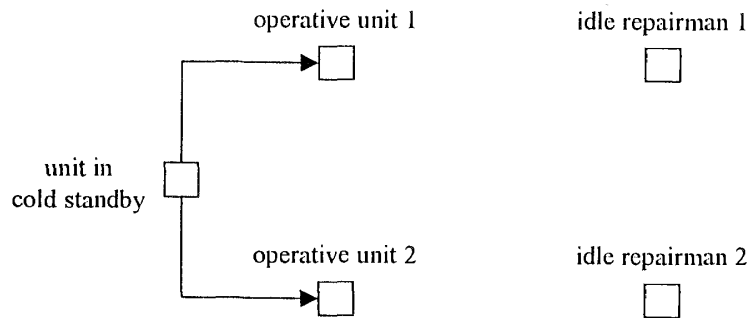

Fig. 1. Functional block-diagram of the T-system operating in state A.

$\{N_t = B\}$: "One repairman is busy at time t, i.e., two units are operating in parallel and one unit is in repair." Note that by assumption, both repairmen are statistically equal. Hence, it is by no means necessary to specify which repairman is busy or idle at time t.

Fig. 2 shows a functional block-diagram of the T-system operating in state B.

$\{N_t = C\}$: "Both repairmen are jointly busy and only one unit is operative at time t." Fig. 3 shows a functional block-diagram of the T-system operating in state C.
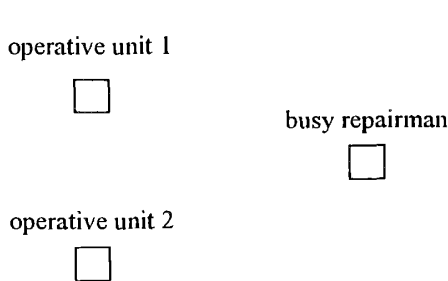

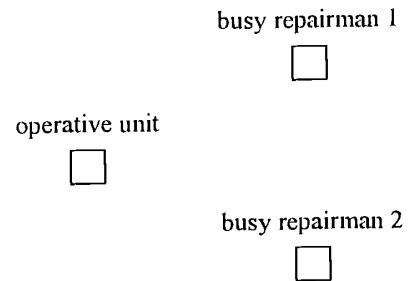Fig. 2. Functional block-diagram of the T-system operating in state B.

Fig. 3. Functional block-diagram of the T-system operating in state C.

$\{N_t = D\}$: "Both repairmen are jointly busy and a failed unit is waiting for repair at time t."

We define the stopping time

$$\theta := \inf\{t > 0 : N_t = D \mid N_0 = A\}.$$

In reliability engineering, θ is usually called the first system-down time. System's survivor function is defined by $S(t) := \mathbf{P}\{\theta > t\}$.

It is plain that the behavior of the process $\{N_t, t \geq 0\}$ *after θ* is irrelevant for the analysis of the first system-down time. Therefore, we consider the system-down state D as an absorbing state.

A Markov characterization of the process $\{N_t, t \geq 0\}$ is piecewise and conditionally defined by:

$\{N_t, t \geq 0\}$, if $N_t = A$ (the renewal state), $N_0 = A$.

$\{(N_t, X_t), t \geq 0\}$, if $N_t = B$, where X_t denotes the elapsed repair time of the failed unit in progress at time t.

$\{(N_t, X_t, Y_t), t \geq 0\}$, if $N_t = C$, where (X_t, Y_t) denotes a random permutation of the elapsed repair times of failed units in progress at time t.

$\{N_t, t \geq 0\}$, if $N_t = D$.

The state space of the process is given by

$$\{A\} \cup \{(B, x); x \geq 0\} \cup \{(C, x, y); x \geq 0, y \geq 0\} \cup \{D\}.$$

Let $p_D(t) := \mathbf{P}\{N_t = D\}$ and for $K = A, B, C$ let

$$p_K(t) := \mathbf{P}\{N_t = K, \forall u : 0 < u < t, N_u \neq D\},$$

where $\forall t \geq 0$, $p_A(t) + p_B(t) + p_C(t) + p_D(t) = 1$.

Finally, we introduce the transition kernels

$$p_B(t, x)dx := \mathbf{P}\{N_t = B, \forall u : 0 < u < t, N_u \neq D, x < X_t \leq x + dx\}.$$

$$p_C(t, x, y)dxdy := \mathbf{P}\{N_t = C, \forall u : 0 < u < t, N_u \neq D, x < X_t \leq x + dx, y < Y_t \leq y + dy\}.$$

Note that $p_D(t) = \mathbf{P}\{\theta \leq t\}, t \geq 0$.

3. Integro-differential Equation

In order to construct a set of differential equations, we apply a time dependent version of the supplementary variable method. For $t > 0$, respectively $t > x > 0$, we obtain the Kolmogorov-type equations

$$(2\lambda + \frac{d}{dt})p_A(t) = \int_0^t p_B(t, x)\frac{dR(x)}{R^-(x)} \tag{1}$$

$$(2\lambda + \frac{1}{R^-(x)}\frac{dR}{dx} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x})p_B(t, x) = 2! \int_0^t p_C(t, x, y)\frac{dR(y)}{R^-(y)}. \tag{2}$$

A conditional probabilistic argument reveals that

$$p_C(t, x, y) = \begin{cases} p_C(t - y, x - y, 0)e^{-\lambda y}\dfrac{R^-(x)R^-(y)}{R^-(x - y)}, & \text{if } t \geq x \geq y \geq 0, \\ p_C(t - x, 0, y - x)e^{-\lambda x}\dfrac{R^-(x)R^-(y)}{R^-(y - x)}, & \text{if } t \geq y \geq x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since D is an absorbing state, we have

$$p_D(t) = \lambda \int_0^t p_C(u)du.$$

The boundary conditions are

$$p_A(0) = 1, \quad p_B(t,0) = 2\lambda p_A(t), \quad 2!p_C(t,x,0) = \begin{cases} 2\lambda p_B(t,x), & \text{if } t \geq x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In order to simplify the equations, let

$$\Phi(u,v) := \begin{cases} p_C(u,v,0)\dfrac{1}{R^-(v)}, & \text{if } u \geq v \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

But note that $p_C(u,v,0) = p_C(u,0,v)$. Clearly,

$$p_B(t,x) = \begin{cases} \lambda^{-1}\Phi(t,x)R^-(x), & \text{if } t \geq x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Whence, by equation (1),

$$(2\lambda + \frac{d}{dt})p_A(t) = \lambda^{-1} \int_0^t \Phi(t,y)dR(y). \tag{3}$$

Moreover,

$$p_B(t) = \lambda^{-1} \int_0^t \Phi(t,y)R^-(y)dy. \tag{4}$$

In order to simplify the integro-differential equation (2), we simply remark that on the one hand,

$$(2\lambda + \frac{1}{R^-(x)}\frac{dR}{dx} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x})p_B(t,x) = \lambda^{-1}R^-(x)(2\lambda + \frac{\partial}{\partial t} + \frac{\partial}{\partial x})\Phi(t,x).$$

While on the other hand,

$$\int_0^t p_C(t,x,y)\frac{dR(y)}{R^-(y)} = R^-(x)\int_{y=0}^x \Phi(t-y, x-y)e^{-\lambda y}dR(y)$$
$$+ R^-(x)e^{-\lambda x}\int_{y=x}^t \Phi(t-x, y-x)dR(y).$$

So that,

$$(2\lambda + \frac{\partial}{\partial t} + \frac{\partial}{\partial x})\Phi(t,x) = 2\lambda \int_0^x \Phi(t-y, x-y)e^{-\lambda y}dR(y) \tag{5}$$
$$+ 2\lambda e^{-\lambda x}\int_{y=x}^t \Phi(t-x, y-x)dR(y).$$

Observe that $\Phi(t,0) = 2\lambda^2 p_A(t)$, $t \geq 0$.

Laplace transforms of functions, with respect to t, are denoted by the corresponding character marked with an asterisk. For instance,

$$\Phi^*(s,u) := \int_{t=0}^\infty \Phi(t,u)e^{-st}dt, \quad s > 0, u \geq 0.$$

But note that

$$\Phi^*(s,u) = \int_{t=u}^\infty \Phi(t,u)e^{-st}dt.$$

A Laplace transform technique, sustained by Fubini's theorem and Leibnitz's differentiation formula, applied to equations (3), (4) and (5) (see Appendix 1) yields the following

Result For $s > 0$,

$$p_A^*(s) = \frac{1}{s + 2\lambda(1 - \int_0^\infty \Psi^*(s,z)dR(z))}, \qquad p_B^*(s) = \frac{2\lambda \int_0^\infty \Psi^*(s,z)R^-(z)dz}{s + 2\lambda(1 - \int_0^\infty \Psi^*(s,z)dR(z))},$$

where $\Psi^*(s,x)$, $x > 0$, satisfies the integro-differential equation

$$(s + 2\lambda + \frac{d}{dx})\Psi^*(s,x) = 2\lambda \int_0^x \Psi^*(s,x-y)e^{-(s+\lambda)y}dR(y) \tag{6}$$

$$+ 2\lambda e^{-(s+\lambda)x}\int_{y=x}^\infty \Psi^*(s,y-x)dR(y)$$

with boundary condition $\Psi^*(s,0) = 1$.

4. The Survivor Function

From the relation

$$p_D(t) = \lambda \int_0^t p_C(u)du,$$

we obtain

$$Ee^{-s\theta} = \lambda p_C^*(s).$$

However, the evaluation of $p_C^*(s)$ as a functional of $\Psi^*(s, \bullet)$ is quite complicated. Therefore, in order to derive a simpler formula, we invoke the system of equations

$$p_A(t) + p_B(t) + p_C(t) + p_D(t) = 1, \qquad \lambda^{-1}\frac{d}{dt}p_D(t) = p_C(t).$$

Eliminating $p_C(t)$, reveals that

$$Ee^{-s\theta} = \frac{\lambda}{\lambda + s}(1 - sp_A^*(s) - sp_B^*(s)).$$

Note that $S(t)$ is uniquely determined by the Laplace transform

$$\frac{1 - Ee^{-s\theta}}{s}, \quad s > 0.$$

5. Solution Procedure

In order to introduce the solution procedure, we first transform the Fredholm equation into a Cauchy integral equation.

For Re $\omega \geq 0$, let

$$\wp(s,\omega) := \int_0^\infty \Psi^*(s,x)e^{-\omega x}dx, \qquad r(\omega) := \int_0^\infty e^{-\omega x}dR(x).$$

Applying Fubini's theorem and Hewitt's inversion formula to equation (6) (see Appendix 2) reveals that

$$[s + 2\lambda(1 - r(s + \lambda + \omega)) + \omega]\wp(s,w) = 1 + 2\lambda\frac{1}{2\pi i}\oint \wp(s,\eta)r(-\eta)\frac{d\eta}{\eta + s + \lambda + \omega}. \tag{7}$$

Remark   It is possible to prove that the Cauchy integral equation also holds for an arbitrary repair time distribution. Unfortunately, the solution of the equation is, in general, extremely formal (cf. [15, pp. 496–497]).

In order to present computational results, we outline the solution procedure for a repair time distribution with an *arbitrary* rational Laplace-Stieltjes transform, i.e. let

$$r(\omega) = \frac{Q_m(\omega)}{Q_n(\omega)},$$

where $Q_p(\omega)$, $p = n, m$; $m < n$ is a polynomial of degree p.

Cox [3] has shown that this family is surprisingly large.

Applying the residue theorem yields

$$\frac{1}{2\pi i} \oint \wp(s,\eta) r(-\eta) \frac{d\eta}{\eta + \lambda + s + \omega} = \sum_k \operatorname*{Res}_{z=\lambda_k} \{ \wp(s,z) \frac{Q_m(-z)}{Q_n(-z)} \frac{-1}{z + \lambda + s + \omega} \},$$

where $\{\lambda_k, \operatorname{Re}\lambda_k > 0; k = 1, 2, \cdots, n\}$ are the n roots of the equation $Q_n(-z) = 0$.

Note that the "first" singularity, i.e. the root nearest to the origin, is always real [16].

Evaluation of the sum by the methods of residues (called elementary operations) yields a weighted mixture of rational transforms. The "weights" are functionals of the form

$$\left(\frac{\partial}{\partial z} \right)^k \wp(s,z) \mid_{z=\lambda_j}; k = 0, 1, \cdots, K_j - 1,$$

where K_j is the multiplicity of the root λ_j.

The functionals are called operating characteristics [13]. Further elementary operations (characterized by the multiplicity of the roots) applied to the equation

$$[s + 2\lambda(1 - \frac{Q_m(s + \lambda + \omega)}{Q_n(s + \lambda + \omega)}) + \omega]\wp(s,\omega) = 1 + 2\lambda \sum_k \operatorname*{Res}_{z=\lambda_k} \{ \wp(s,z) \frac{Q_m(-z)}{Q_n(-z)} \frac{-1}{z + \lambda + s + \omega} \},$$

yields a system of n linear equations in the n unknown operating characteristics. The solution of the corresponding matrix equation determines $\wp(s,\omega)$ uniquely. However, it is by no means necessary to invert $\wp(s,\omega)$. As a matter of fact, the required integrals

$$\int_0^\infty \Psi^*(s,x) dR(x) \quad \text{and} \quad \int_0^\infty \Psi^*(s,x) R^-(x) dx,$$

are easily evaluated in terms of the operating characteristics! So that $Ee^{-s\theta}$ is completely determined.

6.   Numerical Example

Let

$$R(t) = \sum_{k=1}^n p_k(1 - e^{-\lambda_k t}), \ n \geq 1, \ \lambda_k > 0, \ \sum_{k=1}^n p_k = 1,$$

where, without loss of generality, $\lambda_1 < \cdots < \lambda_n$.

We do *not* require that all p_k are positive (which is the case in the family of hyper-exponentials). As $R(t)$ is supposed to be a probability distribution of a positive random variable, we must have

$$p_1 > 0; \quad \sum_{k=1}^n p_k \lambda_k \geq 0.$$

Note that, for instance,

$$(R^-(t))^{-1}; \quad n = 2, \ p_1 > 0, \ p_2 < 0, \ p_1 + p_2 = 1, \ \lambda_1 p_1 + \lambda_2 p_2 \geq 0,$$

is log-convex, so that R belongs to the important family of repair time distributions with an increasing repair rate [14]. Clearly,

$$r(\omega) = \sum_{k=1}^{n} p_k \frac{\lambda_k}{\lambda_k + \omega}.$$

Whence,

$$[s + 2\lambda(1 - \sum_{k=1}^{n} p_k \frac{\lambda_k}{s + \lambda + \lambda_k + \omega}) + \omega]\wp(s, \omega) = 1 + 2\lambda \sum_{k=1}^{n} \wp(s, \lambda_k) p_k \frac{\lambda_k}{s + \lambda + \lambda_k + \omega}. \quad (8)$$

Inserting $\omega = \lambda_j$; $j = 1, 2, \cdots, n$ into equation (8), yields the system of n linear equations,

$$[s + 2\lambda(1 - \sum_{k=1}^{n} p_k \frac{\lambda_k}{s + \lambda + \lambda_k + \lambda_j}) + \lambda_j]\wp(s, \lambda_j) = 1 + 2\lambda \sum_{k=1}^{n} \wp(s, \lambda_k) p_k \frac{\lambda_k}{s + \lambda + \lambda_k + \lambda_j}$$

in the n operating characteristics $\{\wp(s, \lambda_j); j = 1, \cdots, n\}$. Observe that

$$\int_0^\infty \Psi^*(s, x) dR(x) = \sum_{k=1}^{n} p_k \lambda_k \wp(s, \lambda_k), \quad \int_0^\infty \Psi^*(s, x) R^-(x) dx = \sum_{k=1}^{n} p_k \wp(s, \lambda_k).$$

Hence,

$$Ee^{-s\theta} = \frac{\lambda}{\lambda + s}(1 - s\frac{1 + 2\lambda \sum_{k=1}^{n} \wp(s, \lambda_k) p_k}{s + 2\lambda[1 - \sum_{k=1}^{n} \wp(s, \lambda_k) p_k \lambda_k]}).$$

It is not hard to verify that $Ee^{-s\theta}$ is a rational transform.

Consequently, the inversion procedure requires the solution of a polynomial equation. Apart from the (uninteresting) case $n = 1$, the degree of our polynomial equation is not less than six. According to Abel's theorem, a *general* polynomial equation of degree $d \geq 5$ is not solvable by squarerooting. So that, in general, numerical methods (e.g. [17]) are at order.

Let, for instance, $n = 2$; $\lambda = 0.1$; $\lambda_1 = 2$; $\lambda_2 = 20$; $p_1 = 1.1$; $p_2 = -0.1$. Numerical inversion of the Laplace transform

$$\frac{1 - Ee^{-s\theta}}{s}, \quad \text{Re } s > 0$$

yields

$$S(t) = 1.00037e^{-0.000518651t} - 4.69399 \ 10^{-4}e^{-2.03573t} + 9.69742 \ 10^{-5}e^{-4.50393t}$$
$$+4.84331 \ 10^{-7}e^{-20.0075t} - 4.38906 \ 10^{-7}e^{-22.2926t} + 1.29757 \ 10^{-9}e^{-40.0597t}.$$

Figure 4 displays the graph of $S(t)$ restricted to the interval I that ensures a required reliability level of at least 95%.
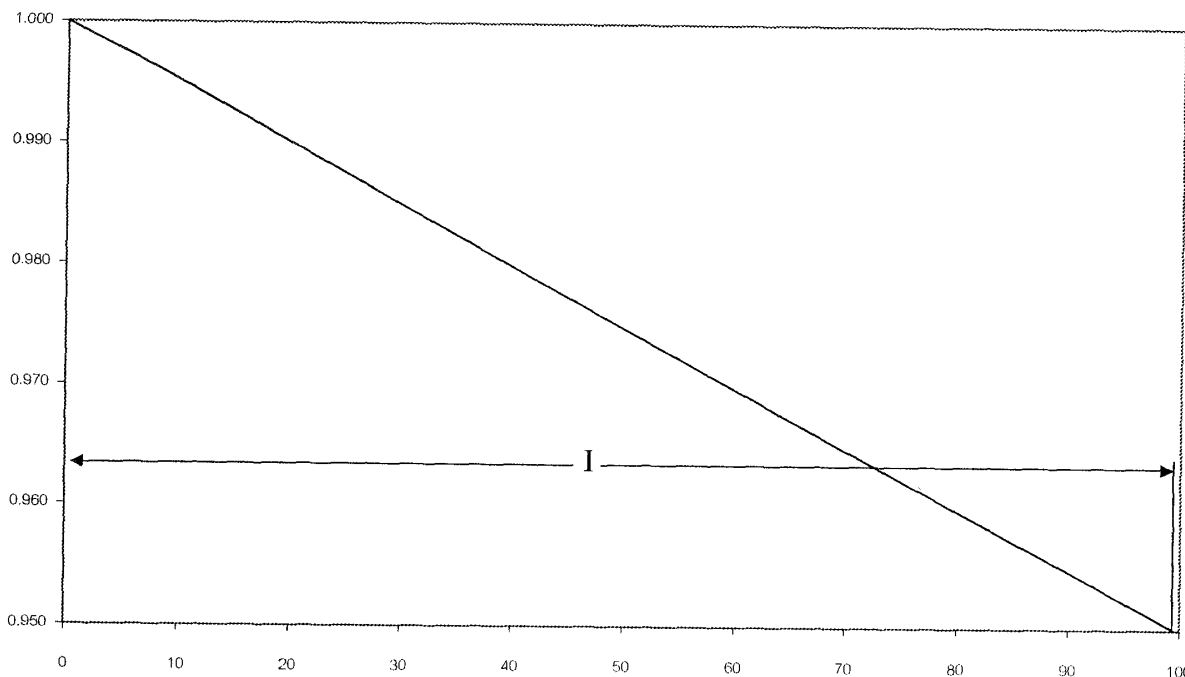
Fig. 4. Graph of $S(t)$ restricted to the security interval I.

7. Conclusion

The reliability analysis of Gaver's parallel system sustained by a cold standby unit and attended by two (statistically) identical repairmen requires the introduction of an appropriate stochastic process endowed with probability kernels satisfying general Kolmogorov-type equations.

The basic function satisfies the Fredholm integro-differential equation

$$(s + 2\lambda + \frac{d}{dx})\Psi^*(s, x) = 2\lambda \int_0^x \Psi^*(s, x - y)e^{-(s+\lambda)y}dR(y)$$

$$+2\lambda e^{-(s+\lambda)x} \int_{y=x}^{\infty} \Psi^*(s, y - x)dR(y).$$

In order to avoid the intricate technical manipulations related to convolutions, we have transformed the Fredholm equation into a Cauchy-type integral equation. But the solution is, in general, extremely formal. However, the solution procedure for a repair time distribution with an arbitrary rational Laplace-Stieltjes transform (a so-called Coxian distribution) is fairly simple and quite suitable for a standard computer routine.

Finally, we remark that repair time distributions with a complicated LST (such as the log-normal and Weibull distribution) could be substituted by suitable approximations composed of Coxian distributions, e.g. [15, Ref 32].

Consequently, a concatenation of our procedure and the proposed approximation technique provides a powerful tool to solve very general problems in reliability engineering.

Appendix 1

The following technical manipulations are justified by Fubini's theorem and Leibnitz's differentiation formula:

$$\int_{t=0}^{\infty} e^{-st} \int_{u=0}^{t} \Phi(t, u)dR(u)dt = \int_{u=0}^{\infty} \int_{t=u}^{\infty} e^{-st}\Phi(t, u)dtdR(u).$$

$$\int_{t=0}^{\infty} e^{-st} \int_{y=0}^{x} \Phi(t-y, x-y)e^{-\lambda y}dR(y)dt = \int_{y=0}^{x} \int_{t=y}^{\infty} e^{-s(t-y)}\Phi(t-y, x-y)dt e^{-(s+\lambda)y}dR(y)$$

$$= \int_{y=0}^{x} \int_{z=0}^{\infty} e^{-sz}\Phi(z, x-y)dz e^{-(s+\lambda)y}dR(y).$$

$$\int_{t=0}^{\infty} e^{-st} \int_{y=x}^{\infty} \Phi(t-x, y-x)dR(y)dt = e^{-sx} \int_{y=x}^{\infty} \int_{t=x}^{\infty} e^{-s(t-x)}\Phi(t-x, y-x)dt dR(y)$$

$$= e^{-sx} \int_{y=x}^{\infty} \int_{z=0}^{\infty} e^{-sz}\Phi(z, y-x)dz dR(y).$$

$$\int_{t=x}^{\infty} e^{-st} \frac{\partial}{\partial t}\Phi(t, x)dt = s \int_{t=x}^{\infty} e^{-st}\Phi(t, x)dt - e^{-sx}\Phi(x, x). \quad \text{(by partial integration)}$$

$$\int_{t=x}^{\infty} e^{-st} \frac{\partial}{\partial x}\Phi(t, x)dt = \frac{\partial}{\partial x} \int_{t=x}^{\infty} e^{-st}\Phi(t, x)dt + e^{-sx}\Phi(x, x). \quad \text{(by Leibnitz)}$$

By definition of $\Phi^*(s, x)$ and the preceding operations, we obtain for $s > 0$,

$$(s + 2\lambda)p_A^*(s) = 1 + \lambda^{-1} \int_0^{\infty} \Phi^*(s, y)dR(y), \quad p_B^*(s) = \lambda^{-1} \int_0^{\infty} \Phi^*(s, y)R^-(y)dy,$$

where $\Phi^*(s, x)$, $x > 0$, satisfies the equation

$$(s + 2\lambda + \frac{d}{dx})\Phi^*(s, x) = 2\lambda \int_{y=0}^{x} \Phi^*(s, x-y)e^{-(s+\lambda)y}dR(y) + 2\lambda e^{-(s+\lambda)x} \int_{y=x}^{\infty} \Phi^*(s, y-x)dR(y),$$

with boundary condition $\Phi^*(s, 0) = 2\lambda^2 p_A^*(s)$.

Finally, invoking the obvious substitution

$$\Phi^*(s, u) = 2\lambda^2 p_A^*(s)\Psi^*(s, u), \quad u \geq 0,$$

yields the announced result.

Appendix 2

The following operations are justified by Fubini's theorem and Hewitt's inversion formula:

$$\int_0^{\infty} e^{-\omega x} \int_0^{x} \Psi^*(s, x-y)e^{-(s+\lambda)y}dR(y)dx = \wp(s, \omega)r(s + \lambda + \omega).$$

$$\int_{x=0}^{\infty} e^{-(\omega+s+\lambda)x} \int_{y=x}^{\infty} \Psi^*(s, y-x)dR(y)dx$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} \Psi^*(s, y-x)e^{-(\omega+s+\lambda)x}dx dR(y) \quad \text{(by Fubini)}$$

$$= \frac{1}{2\pi i}\oint \int_{y=0}^{\infty} e^{-\eta y} \int_{x=0}^{y} \Psi^*(s, y-x)e^{-(\omega+s+\lambda)x}dx dy \int_{y=0}^{\infty} e^{\eta y}dR(y)d\eta \quad \text{(by Hewitt)}$$

$$= \frac{1}{2\pi i}\oint \wp(s, \eta)r(-\eta)\frac{d\eta}{\eta + \lambda + s + \omega}.$$

The symbol $\subset$ denotes the Cauchy principal value of the integral along the imaginary axis. On the other hand,

$$\int_0^{\infty} e^{-\omega x} \frac{d}{dx}\Psi^*(s, x)dx = \omega\wp(s, \omega) - 1.$$

Whence,

$$[s + 2\lambda(1 - r(s + \lambda + \omega)) + \omega]\wp(s, \omega) = 1 + 2\lambda\frac{1}{2\pi i}\oint \wp(s, \eta)r(-\eta)\frac{d\eta}{\eta + s + \lambda + \omega}.$$

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Edmond J. Vanderperre
Faculty of Information Technology
King Mongkut's Institute of Technology Ladkrabang
10520 Bangkok, Thailand