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APPLICATION OF SMOOTHED PERTURBATION ANALYSIS TO A DISCRETE-TIME STATIONARY QUEUE

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Abstract We investigate the *sensitivity analysis* for a discrete-time queueing system using perturbation analysis. In discrete-time queues, not only the sample performance function but also the sampled event lifetimes are essentially discontinuous in some system parameter, and therefore the well-known infinitesimal perturbation analysis (IPA) technique fails to apply to even a simple single-server queue. We here apply an idea similar to the smoothed perturbation analysis for probabilistic routing problem (or equivalently, negative customer rare perturbation analysis), and under the stationarity and ergodicity assumptions, we obtain the strongly consistent estimator. Some simulation experiments demonstrate the validity of our estimator.

1. Introduction

With the development of digital information and telecommunication systems, discrete-time queueing systems have been actively studied in these days (cf. Miyazawa and Takagi eds. [22]). In this paper, we consider the sensitivity analysis for a discrete-time queueing system. For continuous-time discrete event stochastic systems, a variety of techniques for sensitivity estimation have been proposed and studied in this decade (see e.g. Glasserman [12], and Rubinstein and Shapiro [25]). Among them, the infinitesimal perturbation analysis (IPA) is known as the most basic and efficient method, but in its application, it requires the strict condition on the continuity and differentiability of the sample performance function ([12]). To cover a broader class of performance measures and systems, smoothed perturbation analysis (SPA) has been introduced by Gong and Ho [15] and further extended by Fu and Hu [9, 10]. The likelihood ratio/score function (LR/SF) method is also applicable for broader class of systems, but it is known that the variance of the estimate grows along with the run length of the simulation and hence this method requires relatively short regeneration periods ([25]). Another powerful method is the rare perturbation analysis (RPA); the basic idea of this approach is to compare two *rarely different* processes, which are not infinitesimally close when different, by considering the infinitesimal rate of change in the processes. This technique was originally developed as a kind of SPA for the derivative estimation with respect to the admission probability of a queue with routing control (Gong [14]) or as a negative/phantom RPA for derivative with respect to the rate of a Poisson arrival process (Brémaud and Vázquez-Abad [7]). And further developments include the virtual customer RPA (Baccelli and Brémaud [1]) and maximal coupling RPA (Brémaud [3] and Brémaud and Massoulié [6]). Brémaud and Gong [4] give the unified view and the relation of these sensitivity estimation techniques (see also the recent monograph by Fu and Hu [11]).

When we try to apply the PA techniques to discrete-time systems, we are confronted with the problem that not only the sample performance function but also the sampled event lifetimes are essentially discontinuous in the continuous parameter of interest. Therefore, the IPA technique fails to apply to even a simple single-server queue. In this paper, we treat a discrete-time single-server queue and, using an idea similar to the SPA for probabilistic routing (or equivalently, negative RPA), derive the derivative estimator for the steady-state expectation with respect to the parameter of the service time distribution. We should note that a similar problem about the continuous-time queue with two different service times is considered using structural IPA (SIPA) by Dai and Ho [8] and that a similar idea is also found in Kesidis *et al.* [16]. We proceed within the stationarity framework and, under the ergodicity assumption, obtain the strongly consistent derivative estimator. As for the PA in the stationarity and ergodicity framework, Konstantopoulos and Zazanis [17, 18] and Brémaud and Lasgouttes [5] derive the IPA estimators for the stationary and ergodic $G/G/1/\infty$ queue using the Palm inversion formula, and Miyoshi [24] extends them to the SPA for multiclass queues. Brémaud and Gong [4] give the stationary SPA (RPA) formula for the probabilistic admission control problem of a $G/G/1$ queue.

The rest of this paper is organized as follows: In the next section, the discrete-time queueing model we consider is detailed with introducing some notation, where the formulation is due to the Palm framework (see e.g. Baccelli and Brémaud [2]). In Section 3, under some appropriate assumptions, we derive the derivative expression which is unbiased in the steady state, and in Section 4, we use the ergodic theorem to obtain the strongly consistent estimate. Section 5 contains some results of simulation experiments, which demonstrate the validity of our estimates. Finally, it is slightly discussed about the convergence rate of derivative estimates in Section 6.

2. Model Description

Consider a single-server queue with an unlimited buffer. Let $\{T_n\}_{n\in\mathbb{Z}}$ be a sequence of arrival times of customers to the system, where each T_n takes its value on the set of integers $\mathbb{Z}$. Conventionally, we assume that $\{T_n\}_{n\in\mathbb{Z}}$ satisfies

$$\cdots < T_{-1} < T_0 \le 0 < T_1 < T_2 < \cdots ;$$
$$\lim_{n\to\pm\infty} T_n = \pm\infty.$$

The first property means that the arrival process is *simple*, that is, only one customer arrives at a time, and the second says that there are only finite number of arrivals on a bounded interval. Let $\{\tau_n\}_{n\in\mathbb{Z}}$ denote the interarrival time sequence satisfying $\tau_n = T_{n+1} - T_n$ and let N be the counting measure on $(\mathbb{Z}, \mathcal{B}(\mathbb{Z}))$ with respect to $\{T_n\}_{n\in\mathbb{Z}}$, that is,

$$N(A) = \sum_{n\in\mathbb{Z}} 1_A(T_n) \quad \text{for } A \in \mathcal{B}(\mathbb{Z}).$$

For each $n \in \mathbb{Z}$, the nth arriving customer requires the service of $\sigma_n(\theta)$ in discrete-time units, where θ is a real parameter in an interval $\Theta \subset \mathbb{R}$. We assume that service times are independent, identically distributed (i.i.d.) and independent of the arrival process. The server attends to one unit of work (if any) in a unit length of time. Let $G(\cdot,\, \theta)$ on $\mathbb{R}_+$ be the common distribution function of service times assumed right continuous with left limits. $G(\cdot,\, \theta)$ is piecewise constant and its discontinuities are in the set $\mathcal{L} \subset \mathbb{N} = \{1, 2, \dots\}$. Let $g(x,\, \theta) = G(x,\, \theta) - G(x-,\, \theta)$ for any $x > 0$, that is the jump size of G at x. Note that $g(x,\, \theta)$ takes positive value only on $\mathcal{L}$ and, for each $k \in \mathcal{L}$, $g(k,\, \theta)$ represents the probability that the service time of a customer equals to k. In order to define the probability space independent of the parameter θ, we introduce the inverse function of $G(\cdot,\,\theta)$ on $[0,\,1]$, as usual in the perturbation analysis literature,

$$G^{-1}(u,\,\theta) = \inf\{x > 0 : G(x,\,\theta) \ge u\}.$$

Applying the sequence $\{U_n\}_{n\in\mathbb{Z}}$ of independent and uniformly distributed random variables on $[0,\,1]$, also independent of $\{T_n\}_{n\in\mathbb{Z}}$, the nth service time is then given by $\sigma_n(\theta) = G^{-1}(U_n,\,\theta)$ and is indeed distributed according to $G(\cdot,\,\theta)$. By the definition, $\sigma_n(\theta)$ takes its value on $\mathcal{L}$. Choosing the sample space Ω such that a sample point $\omega = \{(T_n,\,U_n)\}_{n\in\mathbb{Z}}$, we define the probability space $(\Omega,\,\mathcal{F},\,\mathrm{P})$ independent of θ. Let $\{\vartheta_i\}_{i\in\mathbb{Z}}$ be the family of measurable shift operators on $(\Omega,\,\mathcal{F})$ satisfying

$$\{(T_n,\,U_n)\}_{n\in\mathbb{Z}} \circ \vartheta_i = \{(T_n - i,\,U_n)\}_{n\in\mathbb{Z}} \quad \text{for any } i \in \mathbb{Z},$$

and we assume that the sequence $\{(T_n,\,U_n)\}_{n\in\mathbb{Z}}$ forms a *discrete-time stationary marked point process*, that is,

$$\mathrm{P} \circ \vartheta_i = \mathrm{P} \quad \text{for any } i \in \mathbb{Z}.$$

We further assume that $(N,\,\mathrm{P})$ has a nonzero intensity $\lambda = \mathrm{E}[N(\{0\})] = \mathrm{P}(T_0 = 0)$ and that $(\mathrm{P},\,\vartheta_i)$ is ergodic. We note by P^0 and E^0 the Palm probability with respect to $(N,\,\mathrm{P})$ and the corresponding expectation. In the case of simple discrete-time point processes, it clearly follows that $\mathrm{P}^0(A) = \mathrm{P}(A \mid T_0 = 0)$ for any $A \in \mathcal{F}$. Note that $\{(\tau_n,\,U_n)\}_{n\in\mathbb{Z}}$ is compatible with $\{\vartheta_{T_n}\}_{n\in\mathbb{Z}}$, that is, $(\tau_n,\,U_n) = (\tau_0,\,U_0)\circ\vartheta_{T_n}$, and P^0 is invariant with respect to ϑ_{T_n}, $n \in \mathbb{Z}$. Now, in order to apply the perturbation analysis, we impose the following assumption on the service time distribution:

Assumption 1 (i) For any $x \in \mathbb{R}_+$, $G(x,\theta)$ is differentiable and Lipschitz in θ, that is, there exists $K^g(\cdot)$ such that, for any $\theta_1,\,\theta_2 \in \Theta$,

$$\big|G(x,\theta_1) - G(x,\theta_2)\big| \le K^g(x)\,|\theta_1 - \theta_2|.$$

(ii) Set $\mathcal{L}$ does not depend on the value of θ;

(iii) Let $\sigma_n^* = \sup_{\theta\in\Theta} G^{-1}(U_n,\,\theta)$ for each $n \in \mathbb{Z}$. Then, $\lambda\,\mathrm{E}^0[\sigma_0^*] < 1$.

Assumption 1(i) ensures the appropriate smoothness of the service time distribution with respect to θ, and 1(ii) says that the discontinuities of $G(x,\,\theta)$ are preserved from the perturbation in θ. Assumption 1(iii) leads to the existence of the stationary and a.s. finite work process for the queue with input $\{(T_n,\,\sigma_n^*)\}_{n\in\mathbb{Z}}$ (c.f. Loynes [20]). Since the work process with service times $\{\sigma_n^*\}_{n\in\mathbb{Z}}$ dominates that with $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$, there exists the stationary and a.s. finite work process for any $\theta \in \Theta$. Let $\{W_i(\theta)\}_{i\in\mathbb{Z}}$ be the work process in the system. Note that $W_i(\theta)$ takes the value on $\mathbb{Z}_+$ and, as long as $W_i(\theta) > 0$, it decreases by one time unit at a unit of time between two successive arrivals, that is, the process $\{W_i(\theta)\}_{i\in\mathbb{Z}}$ satisfies $W_i(\theta) = \big(W_{i-1}(\theta) - 1\big)^+ + \sum_{n\in\mathbb{Z}} \sigma_n(\theta)\,1_{\{i=T_n\}}$ for $i \in \mathbb{Z}$, where $x^+ = \max(x,0)$. Our performance measure is the functional $J(\theta) = \mathrm{E}\big[f\big(W_0(\theta)\big)\big]$, where f is a nondecreasing mapping from $\mathbb{Z}_+$ to $\mathbb{R}_+$. In the next section, we intend to derive an unbiased estimator for the derivative $\mathrm{d}J(\theta)/\mathrm{d}\theta$.

3. Derivation of the Estimator

In the most of the following, we focus our interest on the right-hand derivative, that is, for $\varDelta\theta$ (> 0) such that $\theta + \varDelta\theta \in \Theta$, we estimate,

$$\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} = \lim_{\varDelta\theta\downarrow 0} \frac{\mathrm{E}\big[f\big(W_0(\theta+\varDelta\theta)\big) - f\big(W_0(\theta)\big)\big]}{\varDelta\theta}. \tag{3.1}$$

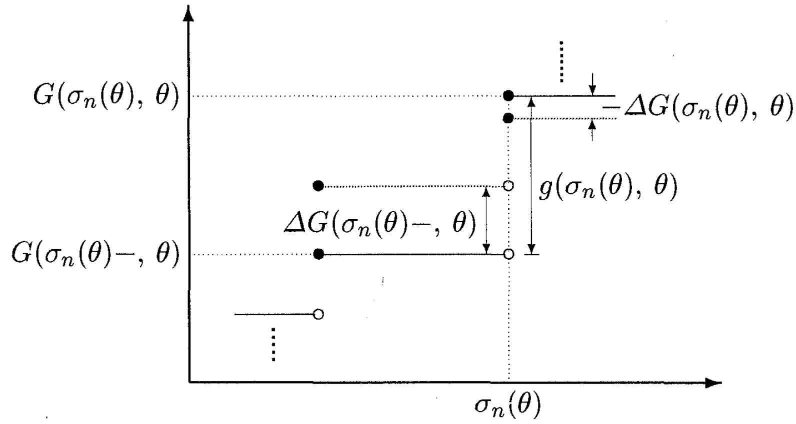


Figure 1: Change of a service time due to a perturbation

The left-hand derivative could be derived in the similar manner. For simplicity of notation, we write $\Delta f\big(W_i(\theta)\big) = f\big(W_i(\theta+\Delta\theta)\big) - f\big(W_i(\theta)\big)$ for each $i \in \mathbb{Z}$, and similarly, for a generic function a of θ, we write $\Delta a(\theta) = a(\theta+\Delta\theta) - a(\theta)$. Let $\{R_n(\theta)\}_{n\in\mathbb{Z}}$ be the sequence of *construction points* at which the arriving customers find the system empty, satisfying $R_0(\theta) \le 0 < R_1(\theta)$. Under the stationarity and Assumption 1(iii), there are infinite number of construction points on the integer line for any $\theta \in \Theta$, and we can keep the realization of the construction points generated by $\{(T_n,\,\sigma_n^*)\}_{n\in\mathbb{Z}}$ in spite of the introduction of $\Delta\theta$, though the additional construction points are possible, that is, noting by $\{R_n^*\}_{n\in\mathbb{Z}}$ the construction points generated by $\{(T_n,\,\sigma_n^*)\}_{n\in\mathbb{Z}}$, we have $\{R_n^*\}_{n\in\mathbb{Z}} \subseteq \{R_n(\theta)\}_{n\in\mathbb{Z}}$ for any $\theta \in \Theta$. Now, we further assume the following:

Assumption 2 (i) Let $\{W_i^*\}_{i\in\mathbb{Z}}$ denote the work process with input $\{(T_n,\,\sigma_n^*)\}_{n\in\mathbb{Z}}$. Then, $\mathrm{E}^0\big[\{\sum_{i=0}^{T_1-1} f(W_i^*)\}^2\big] < \infty$.

(ii) For some $\gamma_1\ (>0)$, $\mathrm{E}^0\big[\exp\{\gamma_1\, N\big([R_0^*,\,R_1^*)\big)\}\big] < \infty$.

(iii) For any $\theta \in \Theta$, $K^g(\cdot)$ in Assumption 1(i) and for some $\gamma_2\ (>0)$, $\sum_{k\in\mathcal{L}} \exp\Big\{\gamma_2\,\dfrac{K^g(k)+K^g(k-)}{g(k,\,\theta)}\Big\}\, g(k,\,\theta) < \infty$.

Assumption 2(ii) ensures the finiteness of any order moments of $N\big([R_0^*,\,R_1^*)\big)$ and (iii) also says that any order moments of $\big\{K^g\big(\sigma_0(\theta)\big) + K^g\big(\sigma_0(\theta)-\big)\big\}/g\big(\sigma_0(\theta),\,\theta\big)$ are finite.

In order to apply the idea of SPA, define the sub-σ-field of $\mathcal{F}_0$ $(= \sigma(\{A \cap \{T_0 = 0\} \mid A \in \mathcal{F}\}))$ by

$$\mathcal{Z}(\theta) = \sigma\Big(\big\{\big(\tau_n,\,\sigma_n(\theta)\big)\big\}_{n\in\mathbb{Z}}\Big).$$

Note that, on $\Omega_0 = \{T_0 = 0\}$, the process $\{W_i(\theta)\}_{i\in\mathbb{Z}}$ is $(\mathcal{Z}(\theta),\,\mathrm{P}^0)$-measurable but $\{W_i(\theta+\Delta\theta)\}_{i\in\mathbb{Z}}$ is not because we can not know whether $\sigma_n(\theta+\Delta\theta) = \sigma_n(\theta)$ or not from the information of $\mathcal{Z}(\theta)$. Before proceeding to our main statement, we consider the conditional probability given $\mathcal{Z}(\theta)$ with which some service times change due to the perturbation in θ. Due to the perturbation of size $\Delta\theta$, each value of $G(x,\,\cdot)$ is shifted by $\Delta G(x,\,\theta)$. Figure 1 illustrates the change of a part of the service time distribution function. By the help of this figure, we see that the conditional probability of the event $\{\sigma_n(\theta+\Delta\theta) \ne \sigma_n(\theta)\}$ given $\sigma_n(\theta)$ is expressed by

$$\mathrm{P}^0\big(\sigma_n(\theta+\Delta\theta) \ne \sigma_n(\theta) \mid \sigma_n(\theta)\big) = \frac{\Delta G\big(\sigma_n(\theta),\,\theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\,\theta\big)^+}{g\big(\sigma_n(\theta),\,\theta\big)} \wedge 1, \qquad (3.2)$$

where $x^- = -\min(x, 0)$ and $a \wedge b = \min(a, b)$. In the above expression (also in Figure 1), $\Delta G(\sigma_n(\theta), \theta)$ means $\Delta G(x, \theta)\big|_{x=\sigma_n(\theta)}$, that is, the Δ-operation does not work for the argument $\sigma_n(\theta)$. We believe that no confusion arises and adopt this abuse of notation hereafter. From Assumptions 1(i) and 2(iii), when $\Delta\theta$ is small enough, we can regard that $\Delta G(\sigma_n(\theta), \theta)^- + \Delta G(\sigma_n(\theta)-, \theta)^+ \le g(\sigma_n(\theta), \theta)$ a.s., and in the rest of this paper, we consider only this case. In such a case, we can say more that

$$\mathrm{P}^0\Big(\sigma_n(\theta+\Delta\theta) = \min\{k \in \mathcal{L} : k > \sigma_n(\theta)\} \mid \sigma_n(\theta)\Big) = \frac{\Delta G(\sigma_n(\theta), \theta)^-}{g(\sigma_n(\theta), \theta)}; \tag{3.3a}$$

$$\mathrm{P}^0\Big(\sigma_n(\theta+\Delta\theta) = \max\{k \in \mathcal{L} : k < \sigma_n(\theta)\} \mid \sigma_n(\theta)\Big) = \frac{\Delta G(\sigma_n(\theta)-, \theta)^+}{g(\sigma_n(\theta), \theta)}. \tag{3.3b}$$

Since the dominant construction points $\{R_n^*\}_{n\in\mathbb{Z}}$ is preserved from the perturbation in θ, the effect to $W_0(\theta)$ due to the perturbation depends on $\Delta\theta$ only through the change of service times of customers arriving during $[R_0^*, 0]$. Let $N_0 = N([R_0^*, 0])$, the number of customers arriving by the time origin from the beginning of the current busy period generated by $\{(T_n, \sigma_n^*)\}_{n\in\mathbb{Z}}$. Note that, using this notation, $R_0^* = T_{-N_0+1}$ and that N_0 is $\mathcal{Z}(\theta)$-measurable for any $\theta \in \Theta$. Let $\mathcal{N}_0 = \{-N_0+1, \dots, 0\}$. And we write for $A \subset \mathcal{N}_0$,

$$\mathcal{D}(\Delta\theta, A; \mathcal{N}_0) = \Big\{\sigma_n(\theta+\Delta\theta) \neq \sigma_n(\theta) \text{ for } n \in A;\ \sigma_m(\theta+\Delta\theta) = \sigma_m(\theta) \text{ for } m \in \mathcal{N}_0\backslash A\Big\}.$$

Then, under the independence of $\{U_n\}_{n\in\mathbb{Z}}$, we have from (3.2),

$$\begin{aligned}\mathrm{P}^0(\mathcal{D}(\Delta\theta, A; \mathcal{N}_0) \mid \mathcal{Z}(\theta)) = &\prod_{n\in A} \frac{\Delta G(\sigma_n(\theta), \theta)^- + \Delta G(\sigma_n(\theta)-, \theta)^+}{g(\sigma_n(\theta), \theta)} \\ &\times \prod_{m\in\mathcal{N}_0\backslash A}\left(1 - \frac{\Delta G(\sigma_m(\theta), \theta)^- + \Delta G(\sigma_m(\theta)-, \theta)^+}{g(\sigma_m(\theta), \theta)}\right),\end{aligned} \tag{3.4}$$

for a sufficiently small $\Delta\theta$. Now, we are at the position to present our main statement of this section:

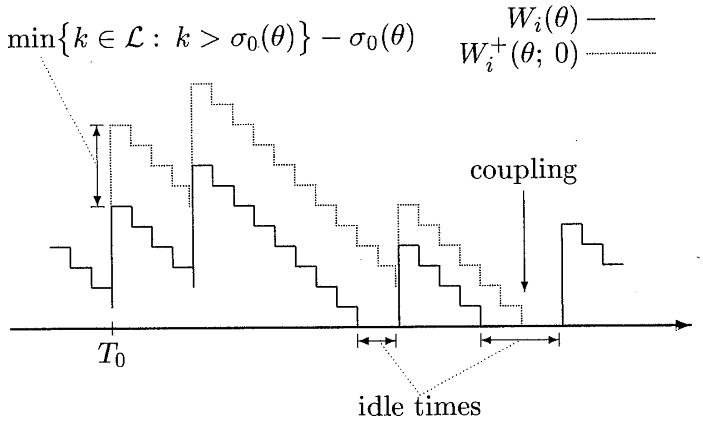
Theorem 1 *Assume that Assumptions 1 and 2 hold. Then,* $J(\theta) = \mathrm{E}\big[f\big(W_0(\theta)\big)\big]$ *admits a right-hand derivative with respect to* θ *given by*

$$\begin{aligned}\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} =& \lambda\, \mathrm{E}^0\Bigg[\frac{\partial_\theta G(\sigma_0(\theta), \theta)^-}{g(\sigma_0(\theta), \theta)} \cdot \sum_{i\in\mathbb{Z}_+} \Delta^+ f(W_i(\theta; 0)) \\ &+ \frac{\partial_\theta G(\sigma_0(\theta)-, \theta)^+}{g(\sigma_0(\theta), \theta)} \cdot \sum_{i\in\mathbb{Z}_+} \Delta^- f(W_i(\theta; 0))\Bigg],\end{aligned} \tag{3.5}$$

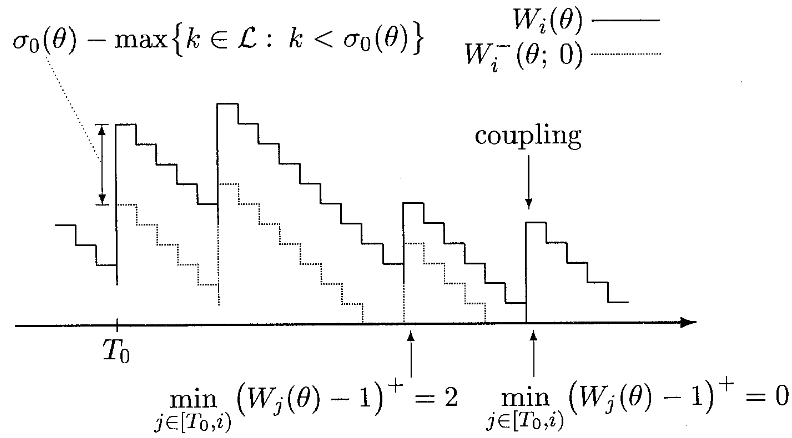
where $\partial_\theta G(\cdot, \theta) = \partial G(\cdot, \theta)/\partial\theta$. *Also,* $\Delta^\pm f\big(W_i(\theta; n)\big) = f\big(W_i^\pm(\theta; n)\big) - f\big(W_i(\theta)\big)$, *and* $W_i^+(\theta; n)$ *[resp.* $W_i^-(\theta; n)$*] represents the value of* $W_i(\theta)$ *given that the service time of the nth customer is* $\min\{k \in \mathcal{L} : k > \sigma_n(\theta)\}$ *[resp.* $\max\{k \in \mathcal{L} : k < \sigma_n(\theta)\}$*] (but if* $\sigma_n(\theta) = \min\{k \in \mathcal{L}\}$*, then* $W_i^-(\theta; n) = W_i(\theta)$*).*

Note that each infinite summation in the right-hand side of (3.5) contains only a finite number of nonzero terms, which suggests the easy implementation of the estimator. Indeed, we have for $i \ge T_0$,

$$W_i^+(\theta; 0) = W_i(\theta) + \Big[\min\{k \in \mathcal{L} :\ k > \sigma_0(\theta)\} - \sigma_0(\theta) - I([T_0, i); \theta)\Big]^+,$$



(a) Sample paths of $W_i(\theta)$ and $W_i^+(\theta;\, 0)$



(b) Sample paths of $W_i(\theta)$ and $W_i^-(\theta;\, 0)$

Figure 2: Relation between the nominal and perturbed sample paths

where $I\big([a,\, b);\, \theta\big) = \sum_{i=a}^{b-1} 1_{\{W_i(\theta)=0\}}$ is the cumulative idle time from a to b ($\sum_{i=a}^{b} = 0$ if $b < a$), and furthermore if $\sigma_0(\theta) \neq \min\{k \in \mathcal{L}\}$,

$$W_i^-(\theta;\, 0) = W_i(\theta) - \min_{j \in [T_0, i)} \big(W_j(\theta) - 1\big)^+ \wedge \Big(\sigma_0(\theta) - \max\{k \in \mathcal{L}:\, k < \sigma_0(\theta)\}\Big),$$

where $\min_{j \in A} = \infty$ if $A = \emptyset$. Thus, both $W_i^+(\theta;\, 0)$ and $W_i^-(\theta;\, 0)$ respectively couple to $W_i(\theta)$ not later than $i = R_1^*$ and $\Delta^{\pm} f\big(W_i(\theta;\, 0)\big)$ vanishes (see Figure 2). The inside of the expectation in (3.5) represents the conditional rate of the service time change of one customer with respect to θ times the effect of such a service time change. Such a form of the estimate is common to the conditional Monte Carlo derivative estimators ([11]).

Proof: Similarly to the proofs for other perturbation analysis formulae, the key tool is the dominated convergence theorem. Applying the discrete-time version of Palm inversion formula and then characterizing by $\mathcal{D}(\Delta\theta,\, A;\, \mathcal{N}_0)$, we have

$$\begin{aligned} \mathrm{E}\big[\Delta f\big(W_0(\theta)\big)\big] &= \lambda\, \mathrm{E}^0\left[\sum_{i=0}^{T_1-1} \Delta f\big(W_i(\theta)\big)\right] \\ &= \lambda\, \mathrm{E}^0\left[\sum_{A \subset \mathcal{N}_0} \sum_{i=0}^{T_1-1} \Delta f\big(W_i(\theta)\big)\, 1_{\mathcal{D}(\Delta\theta,\, A;\, \mathcal{N}_0)}\right], \end{aligned}$$

where the second equality follows since $\mathcal{D}(\Delta\theta,\ A;\ \mathcal{N}_0)$'s are pairwise disjoint. For simplicity of the notation, we write

$$\overline{f_n}(\theta) = \sum_{i=T_n}^{T_{n+1}-1} f\big(W_i(\theta)\big).$$

Since $\mathcal{N}_0$ is $(\mathcal{Z}(\theta),\ \mathrm{P}^0)$-measurable, using the conditional expectation given $\mathcal{Z}(\theta)$,

$$\begin{aligned}\mathrm{E}\big[\Delta f\big(W_0(\theta)\big)\big] &= \lambda\,\mathrm{E}^0\left[\sum_{A\subset\mathcal{N}_0} \mathrm{E}^0\Big[\Delta\overline{f_0}(\theta)\, 1_{\mathcal{D}(\Delta\theta,\,A;\,\mathcal{N}_0)} \mid \mathcal{Z}(\theta)\Big]\right]\\ &= \lambda\,\mathrm{E}^0\left[\sum_{A\subset\mathcal{N}_0} \Delta\overline{f_0}(\theta;\ A)\,\mathrm{P}^0\big(\mathcal{D}(\Delta\theta,\ A;\ \mathcal{N}_0) \mid \mathcal{Z}(\theta)\big)\right],\end{aligned}$$

where $\Delta\overline{f_0}(\theta;\ A)$ represents the value of $\Delta\overline{f_0}(\theta)$ on $\mathcal{D}(\Delta\theta,\ A;\ \mathcal{N}_0)\cap\mathcal{Z}(\theta)$. Note that, once $\mathcal{D}(\Delta\theta,\ A;\ \mathcal{N}_0)$ is given with a sufficiently small $\Delta\theta$, the value of $\Delta\overline{f_0}(\theta;\ A)$ no longer depends on the size of $\Delta\theta$ since the service times of customers with indices in A change only to the possible adjacent values (see (3.3)). This is why the symbol Δ still remains in (3.5) even after $\Delta\theta\to 0$. Using (3.4) for sufficiently small $\Delta\theta$, the inside of the expectation leads to

$$\begin{aligned}&\sum_{A\subset\mathcal{N}_0} \Delta\overline{f_0}(\theta;\ A)\,\mathrm{P}^0\big(\mathcal{D}(\Delta\theta,\ A;\ \mathcal{N}_0) \mid \mathcal{Z}(\theta)\big)\\
&= \sum_{A\subset\mathcal{N}_0} \Delta\overline{f_0}(\theta;\ A) \prod_{n\in A} \frac{\Delta G\big(\sigma_n(\theta),\ \theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\ \theta\big)^+}{g\big(\sigma_n(\theta),\ \theta\big)}\\
&\qquad \times \prod_{m\in\mathcal{N}_0\setminus A}\left(1 - \frac{\Delta G\big(\sigma_m(\theta),\ \theta\big)^- + \Delta G\big(\sigma_m(\theta)-,\ \theta\big)^+}{g\big(\sigma_m(\theta),\ \theta\big)}\right)\\
&= \sum_{n\in\mathcal{N}_0} \Delta\overline{f_0}(\theta;\ \{n\})\,\frac{\Delta G\big(\sigma_n(\theta),\ \theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\ \theta\big)^+}{g\big(\sigma_n(\theta),\ \theta\big)}\\
&\quad - \sum_{n\in\mathcal{N}_0} \Delta\overline{f_0}(\theta;\ \{n\})\,\frac{\Delta G\big(\sigma_n(\theta),\ \theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\ \theta\big)^+}{g\big(\sigma_n(\theta),\ \theta\big)}\\
&\qquad \times \left\{1 - \prod_{m\in\mathcal{N}_0\setminus\{n\}}\left(1 - \frac{\Delta G\big(\sigma_m(\theta),\ \theta\big)^- + \Delta G\big(\sigma_m(\theta)-,\ \theta\big)^+}{g\big(\sigma_m(\theta),\ \theta\big)}\right)\right\}\\
&\quad + \sum_{\substack{A\subset\mathcal{N}_0\\ |A|\ge 2}} \Delta\overline{f_0}(\theta;\ A) \prod_{n\in A} \frac{\Delta G\big(\sigma_n(\theta),\ \theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\ \theta\big)^+}{g\big(\sigma_n(\theta),\ \theta\big)}\\
&\qquad \times \prod_{m\in\mathcal{N}_0\setminus A}\left(1 - \frac{\Delta G\big(\sigma_m(\theta),\ \theta\big)^- + \Delta G\big(\sigma_m(\theta)-,\ \theta\big)^+}{g\big(\sigma_m(\theta),\ \theta\big)}\right).\end{aligned} \tag{3.6}$$

In order to apply the dominated convergence theorem, we find the bounds of the three terms of the last expression of (3.6), respectively. Using the notation of

$$K_0^S(\theta) = \max_{n\in\mathcal{N}_0} \frac{K^g\big(\sigma_n(\theta)\big) + K^g\big(\sigma_n(\theta)-\big)}{g\big(\sigma_n(\theta),\ \theta\big)},$$

we find a bound of the first term as,

$$\begin{aligned}\left|\sum_{n\in\mathcal{N}_0} \Delta\overline{f_0}(\theta;\,\{n\})\,\frac{\Delta G\big(\sigma_n(\theta),\,\theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\,\theta\big)^+}{g\big(\sigma_n(\theta),\,\theta\big)}\right| \\ \leq \Delta\theta\,\overline{f_0^*}\sum_{n\in\mathcal{N}_0}\frac{K^g\big(\sigma_n(\theta)\big)+K^g\big(\sigma_n(\theta)-\big)}{g\big(\sigma_n(\theta),\,\theta\big)} \\ \leq \Delta\theta\,\overline{f_0^*}\,N\big([R_0^*,\,R_1^*)\big)\,K_0^S(\theta),\end{aligned}$$

where $\overline{f_0^*} = \sum_{i=0}^{T_1-1} f(W_i^*)$. From the Schwarz inequality,

$$\mathrm{E}^0\Big[\overline{f_0^*}\,N\big([R_0^*,\,R_1^*)\big)\,K_0^S(\theta)\Big] \leq \sqrt{\mathrm{E}^0\big[\overline{f_0^*}^2\big]}\,\sqrt{\mathrm{E}^0\big[N([R_0^*,\,R_1^*))^2\,K_0^S(\theta)^2\big]},$$

and the first term of the right-hand side is finite from Assumption 2(i). The same procedure leads to

$$\mathrm{E}^0\Big[\big(N([R_0^*,\,R_1^*))^2\,K_0^S(\theta)\big)^2\Big] \leq \sqrt{\mathrm{E}^0\big[N\big([R_0^*,\,R_1^*)\big)^4\big]}\,\sqrt{\mathrm{E}^0\big[K_0^S(\theta)^4\big]},$$

and the both terms are finite from Assumption 2(ii) and (iii), respectively. For the second and third terms of (3.6), we have

$$\begin{aligned}\big|\text{2nd term of (3.6)}\big| &\leq N_0\,\overline{f_0^*}\,K_0^S(\theta)\,\Delta\theta\,\big\{1-\big(1-K_0^S(\theta)\,\Delta\theta\big)^{N_0-1}\big\} \\ &\leq \overline{f_0^*}\,N_0\,(N_0-1)\,K_0^S(\theta)^2\,\Delta\theta^2; \\ \big|\text{3rd term of (3.6)}\big| &\leq \overline{f_0^*}\sum_{l=2}^{N_0}\binom{N_0}{l}\big(K_0^S(\theta)\,\Delta\theta\big)^l.\end{aligned}$$

Therefore, from Assumption 2, the expectations of the absolute values of the second and third terms of (3.6) are both bounded by $o(\Delta\theta)$. Hence, we have

$$\frac{\mathrm{E}\big[\Delta f\big(W_0(\theta)\big)\big]}{\Delta\theta} = \lambda\,\mathrm{E}^0\left[\sum_{n\in\mathcal{N}_0}\frac{\Delta\overline{f_0}(\theta;\,\{n\})}{\Delta\theta}\,\frac{\Delta G\big(\sigma_n(\theta),\,\theta\big)^- + \Delta G\big(\sigma_n(\theta)-,\,\theta\big)^+}{g\big(\sigma_n(\theta),\,\theta\big)}\right] + \frac{o(\Delta\theta)}{\Delta\theta}.$$

Now, applying the dominated convergence theorem and exploiting the ϑ_{T_n}-invariance property of P^0, we have

$$\begin{aligned}\frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta} &= \lambda\,\mathrm{E}^0\left[\sum_{n\in\mathcal{N}_0}\Delta\overline{f_0}(\theta;\,\{n\})\,\frac{\partial_\theta G\big(\sigma_n(\theta),\,\theta\big)^- + \partial_\theta G\big(\sigma_n(\theta)-,\,\theta\big)^+}{g\big(\sigma_n(\theta),\,\theta\big)}\right] \\ &= \lambda\,\mathrm{E}^0\left[\sum_{n\in\mathbb{Z}}\Delta\overline{f_{-n}}(\theta;\,\{0\})\,\frac{\partial_\theta G\big(\sigma_0(\theta),\,\theta\big)^- + \partial_\theta G\big(\sigma_0(\theta)-,\,\theta\big)^+}{g\big(\sigma_0(\theta),\,\theta\big)}\,1_{[0,R_1^*)}(T_{-n})\right],\end{aligned}$$

where we use that $0 \in [\max\{R_m^* \leq T_{-n}\},\,T_{-n}]$ implies $T_{-n} \in [0,\,R_1^*)$. Finally, splitting $\Delta\overline{f_{-n}}(\theta;\,\{0\})$ according to (3.3) and taking the statement about infinite summation under Theorem 1 into account, we obtain (3.5). □

In entirely the same way, we have the left-hand derivative version of Theorem 1:

Theorem 1' *Under the same assumptions as Theorem 1, we have*

$$\frac{\mathrm{d}^- J(\theta)}{\mathrm{d}\theta} = -\lambda\,\mathrm{E}^0\left[\frac{\partial_\theta G(\sigma_0(\theta),\,\theta)^+}{g(\sigma_0(\theta),\,\theta)}\cdot\sum_{i\in\mathbb{Z}_+}\Delta^+ f(W_i(\theta;\,0)) + \frac{\partial_\theta G(\sigma_0(\theta)-,\,\theta)^-}{g(\sigma_0(\theta),\,\theta)}\cdot\sum_{i\in\mathbb{Z}_+}\Delta^- f(W_i(\theta;\,0))\right]. \tag{3.7}$$

Remark 1 When mapping f is finite, Assumption 2(i) is replaced by
(i') $\mathrm{E}^0[\tau_0^2] < \infty$.
When not the case, for instance, Assumption 2(i) can be replaced by
(i-a) $\mathrm{E}^0[\tau_0^4] < \infty$,
(i-b) $\mathrm{E}^0\left[f(W_0^*(\theta))^4\right] < \infty$.

Remark 2 (Monotonous $G(\cdot,\,\theta)$) If the distribution function $G(\cdot,\,\theta)$ is nondecreasing in θ, then $\partial_\theta G(\cdot,\,\theta) \geq 0$ and we have

$$\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} = \lambda\,\mathrm{E}^0\left[\frac{\partial_\theta G(\sigma_0(\theta)-,\,\theta)}{g(\sigma_0(\theta),\,\theta)}\cdot\sum_{i\in\mathbb{Z}_+}\Delta^- f(W_i(\theta;\,0))\right]; \tag{3.8a}$$

$$\frac{\mathrm{d}^- J(\theta)}{\mathrm{d}\theta} = -\lambda\,\mathrm{E}^0\left[\frac{\partial_\theta G(\sigma_0(\theta),\,\theta)}{g(\sigma_0(\theta),\,\theta)}\cdot\sum_{i\in\mathbb{Z}_+}\Delta^+ f(W_i(\theta;\,0))\right]. \tag{3.8b}$$

Remark 3 (Multiple Arrival Points) In this paper, we treat only the case of a simple point process, but it would not be difficult to extend the result to the case of multiple point processes by using the Palm probability with respect to such a process in Miyazawa and Takahashi [23].

4. Implementation of the Estimator

In this section, under the ergodicity assumption, we derive the strongly consistent estimator for $\mathrm{d}^+ J(\theta)/\mathrm{d}\theta$ based on Theorem 1. The left-hand derivative version is omitted since it can be obtained in the same way.

Theorem 2 *Assume that Assumptions 1 and 2 hold. Then, under the ergodicity of* $(\mathrm{P},\,\vartheta_i)$,

$$\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} = \lim_{m\to\infty}\frac{1}{T_m}\sum_{n=0}^{m-1}\left\{\frac{\partial_\theta G(\sigma_n(\theta),\,\theta)^-}{g(\sigma_n(\theta),\,\theta)}\cdot\sum_{i\geq T_n}\Delta^+ f(W_i(\theta;\,n)) + \frac{\partial_\theta G(\sigma_n(\theta)-,\,\theta)^+}{g(\sigma_n(\theta),\,\theta)}\cdot\sum_{i\geq T_n}\Delta^- f(W_i(\theta;\,n))\right\} \quad \mathrm{P}\text{-}a.s. \tag{4.1a}$$

Moreover, if $G(\cdot,\,\theta)$ is nondecreasing in θ, then

$$\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} = \lim_{m\to\infty}\frac{1}{T_m}\sum_{n=0}^{m-1}\frac{\partial_\theta G(\sigma_n(\theta)-,\,\theta)}{g(\sigma_n(\theta),\,\theta)}\cdot\sum_{i\geq T_n}\Delta^- f(W_i(\theta;\,n)) \quad \mathrm{P}\text{-}a.s. \tag{4.1b}$$

Proof: For simplicity, we show equation (4.1b) only. From (3.8a), the cross-ergodic theorem gives

$$\frac{\mathrm{d}^+ J(\theta)}{\mathrm{d}\theta} = \lambda \lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} \frac{\partial_\theta G\big(\sigma_n(\theta)-,\, \theta\big)}{g\big(\sigma_n(\theta),\, \theta\big)} \cdot \sum_{i\geq T_n} \Delta^- f\big(W_i(\theta;\, n)\big) \quad \text{P-a.s.}$$

On the other hand,

$$\lambda = \frac{1}{\mathrm{E}^0[\tau_0]} = \lim_{m\to\infty} \frac{m}{\sum_{n=0}^{m-1} \tau_n} = \lim_{m\to\infty} \frac{m}{T_m} \quad \text{P-a.s.,}$$

and the proof is completed. □

Theorem 2 says that we can obtain the strongly consistent estimator for $\mathrm{d}^+ J(\theta)/\mathrm{d}\theta$ from the single-path calculation (as mentioned in the previous section, each infinite summation in (4.1) contains only a finite number of nonzero terms).

5. Simulation Experiments

In order to show the validity of our estimator, we make simulation experiments for some cases. In the following examples, we take f as the identity mapping, that is, $J(\theta) = \mathrm{E}[W_0(\theta)]$. In Example 1, we treat the case where the analytical result is available, and thus we compare the experimental results from our estimates with the analytical results. The analytical values are computed from the formula in Takagi [26]. In Example 2, we treat the case where the arrival stream has self-correlation and compare the experimental results from our estimates with the values from the symmetric *finite difference* (FD) estimates that are calculated by the following;

$$\left(\frac{\mathrm{d}J(\theta)}{\mathrm{d}\theta}\right)_{\mathrm{fd}} = \frac{\big(J(\theta+\Delta\theta)\big)_{\mathrm{est}} - \big(J(\theta-\Delta\theta)\big)_{\mathrm{est}}}{2\,\Delta\theta}, \tag{5.1}$$

where $(\cdot)_{\mathrm{est}}$ represents the estimate from direct simulation. The FD estimator is indeed biased but known as the last resort to obtaining the derivative estimates. We run the simulation for 80,000 busy periods in each sample path and obtain the estimates with 95% confidence intervals taken from 30 independent replications. The experiments are carried out for three different values of traffic intensity ρ $(= \lambda \mathrm{E}^0[\sigma_0(\theta)])$, corresponding to light traffic ($\rho = 0.2$), medium traffic ($\rho = 0.5$) and heavy traffic ($\rho = 0.8$), by adjusting the value of λ for given θ.

Example 1 (*Geo*/*Geo*/1 queue) In this example, the interarrival and service times of customers are both independent and geometrically distributed with the mean $1/\lambda$ and $1/\theta$, respectively, and the interarrival and service time sequences are also independent each other. The value of θ is fixed at 0.5. In this model, for all $n \in \mathbb{Z}$,

$$\mathrm{P}^0(\tau_n = k) = \lambda\,(1-\lambda)^{k-1};$$
$$\mathrm{P}^0(\sigma_n(\theta) = k) = \theta\,(1-\theta)^{k-1},$$

and we obtain the analytical results from [26] as

$$\frac{\mathrm{d}J(\theta)}{\mathrm{d}\theta} = -\frac{\lambda\,(1-\lambda)\,(2\,\theta-\lambda)}{\theta^2\,(\theta-\lambda)^2}.$$

Table 1: Estimates for a *Geo*/*Geo*/1 queue

ρ	$(J(\theta))_{\text{est}}$	$J(\theta)$	$(\mathrm{d}J(\theta)/\mathrm{d}\theta)_{\text{pa}}$	$\mathrm{d}J(\theta)/\mathrm{d}\theta$
0.2	0.4499±0.0011	0.4500	−2.0252±0.0070	−2.0250
0.5	1.4997±0.0058	1.5000	−8.9963±0.0578	−9.0000
0.8	4.7973±0.0229	4.8000	−57.6162±0.5242	−57.6000

On the other hand, in (4.1b) with $\mathcal{L} = \mathbb{N}$ and f being the identity mapping, $\Delta^{-}W_i(\theta;\, n) = -1_{\{\min_{j\in[T_n,i)} W_j(\theta)>1\}}$ for $\sigma_n(\theta) > 1$ and $i \geq T_n$ since the service time of customer n reduces by one time unit, and therefore, our estimate becomes

$$\left(\frac{\mathrm{d}J(\theta)}{\mathrm{d}\theta}\right)_{\text{pa}} = -\frac{1}{T_m}\sum_{n=0}^{m-1}\frac{\sigma_n(\theta)-1}{\theta\,(1-\theta)}\sum_{i\geq T_n} 1_{\{\min_{j\in[T_n,i)} W_j(\theta)>1\}}.$$

In Table 1, we find that our estimates have the good agreement with the analytical results.

Example 2 (TES-*Geo*/*Geo*/1 queue) In this example, the service times of customers are distributed independently and geometrically as in Example 1, but the interarrival times are distributed according to a correlated geometric distribution; that is, using the transform expanded sample (TES) method (Melamed and Hill [21]), we generate an interarrival time sequence $\{\tau_n\}_{n\geq 0}$ that has a marginally geometric distribution with adjacent correlation.

In this model, we adopt the $\text{TES}^{+}(\alpha,\, \phi)$ method parameterized by $0 \leq \alpha \leq 1$ and $-1 \leq \phi \leq 1$. A brief overview of this method is as follows: For any $x \in \mathbb{R}$, let $[x] = \max\{i \in \mathbb{Z} : i \leq x\}$ be the integral part of x, and define $\langle x\rangle = x - [x]$ to be the fractional part. Let η_0 be a uniform random variable on $[0,\, 1)$, and a sequence $\{\eta_n\}_{n\geq 1}$ be of the form

$$\eta_n = \langle \eta_{n-1} - l + \zeta_n \rangle,$$

where $\{\zeta_n\}_{n\geq 1}$ is independent and uniformly distributed sequence on $[0,\, \alpha)^{\infty}$, and l is derived such that

$$\begin{cases} \alpha = l + r; \\ \phi = (r - l)/\alpha. \end{cases}$$

Moreover, to remove the discontinuity of the sequence $\{\eta_n\}_{n\geq 1}$, we introduce a smoothing transformation S_ξ, parameterized by $0 < \xi < 1$, of the form

$$S_\xi(\eta_n) = \begin{cases} \eta_n/\xi & \text{if } \eta_n \in [0,\, \xi); \\ (1-\eta_n)/(1-\xi) & \text{if } \eta_n \in [\xi,\, 1). \end{cases}$$

Finally from $S_\xi(\eta_n)$ we obtain the interarrival time τ_n by inverting the geometric distribution function. In this simulation, we take the value $\alpha = 0.3$, $\phi = 0.5$ and $\xi = 0.5$. The estimator is the same as in Example 1 with the fixed value of $\theta = 0.5$, and we compare the values from our estimator with the values from FD estimator (5.1), where $\Delta\theta$ is taken as 0.025, 0.01 and 0.005 for $\rho = 0.2$, 0.5 and 0.8, respectively. From Table 2, we see that our estimates show good agreement with FD estimates, with much smaller variance.

Table 2: Estimates for a TES-$Geo/Geo/1$ queue

ρ	$(J(\theta))_{\text{est}}$	$(\mathrm{d}J(\theta)/\mathrm{d}\theta)_{\text{pa}}$	$(\mathrm{d}J(\theta)/\mathrm{d}\theta)_{\text{fd}}$
0.2	0.4592±0.0013	−2.1313±0.0084	−2.1234±0.0361
0.5	1.7099±0.0071	−10.6522±0.0704	−10.6675±0.3773
0.8	5.5420±0.0325	−63.7211±0.6471	−64.0941±3.1553

6. Discussion about the Convergence Rate of Estimates

One might be interested in the asymptotic behavior of strongly consistent estimates as the observation length goes to infinity. Unfortunately, the authors have not found any results concerning this in general case. However, in case that the system has a regenerative structure, some results have been so far obtained, where the convergence rate of estimates is often defined as the order of the square root of the *mean square error* with respect to the number of regenerative cycles. Zazanis and Suri [27] show that, for the FD estimators with independent sources, one can achieve the convergence rates of $O(m^{-1/4})$ for the one-sided difference estimate and $O(m^{-1/3})$ for the symmetric difference estimate by choosing the appropriate parameter difference $\Delta\theta$ in (5.1) as a function of m, where m denotes the number of regenerative cycles. They also show that the IPA estimate with regenerative form has the convergence rate of $O(m^{-1/2})$. For the FD estimators with common random numbers (FDC), Glynn [13] obtains the convergence rates of $O(m^{-1/3})$ for the one-sided case with the parameter difference $\Delta\theta = m^{-1/3}$ and $O(m^{-2/5})$ for the symmetric case with $\Delta\theta = m^{-1/5}$, respectively. Also, L'Ecuyer and Perron [19] show that, under the condition for IPA to apply, FDC has the same order of convergence rate as IPA, that is $O(m^{-1/2})$, provided that the size of the parameter difference goes to zero fast enough.

As for our SPA estimate, if the system has the regenerative structure, we can obtain the same convergence rate as that of the IPA estimate in the similar manner. Now, slightly consider the general case by analogy with the regenerative case. Noting the nth summand of (4.1a) by $\Psi_n(\theta)$ for simplicity, the mean square error of the estimate satisfies

$$\begin{aligned}
&\mathrm{E}^0\left[\left(\frac{1}{T_m}\sum_{n=0}^{m-1}\Psi_n(\theta) - \frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta}\right)^2\right] \\
&\le \frac{1}{m^2}\,\mathrm{E}^0\left[\left\{\sum_{n=0}^{m-1}\left(\Psi_n(\theta) - \tau_n\,\frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta}\right)\right\}^2\right] \\
&= \frac{1}{m}\,\mathrm{E}^0\left[\left(\Psi_0(\theta) - \tau_0\,\frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta}\right)^2\right] \\
&\quad + \frac{1}{m^2}\sum_{n=1}^{m-1}(m-n)\,\mathrm{E}^0\left[\left(\Psi_0(\theta) - \tau_0\,\frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta}\right)\cdot\left(\Psi_n(\theta) - \tau_n\,\frac{\mathrm{d}^+J(\theta)}{\mathrm{d}\theta}\right)\right],
\end{aligned}$$

where the inequality holds since $T_m \ge m$ from the simple discrete-time point process property. Since $\mathrm{E}^0\left[\Psi_0(\theta) - \tau_0\,\mathrm{d}^+J(\theta)/\mathrm{d}\theta\right] = 0$ from Theorem 1, we could obtain some convergence rate (not smaller than $O(m^{-1/2})$) by imposing the assumption such as the correlation in $\left\{\Psi_n(\theta) - \tau_n\,\mathrm{d}^+J(\theta)/\mathrm{d}\theta\right\}_{n\in\mathbb{Z}}$ vanishes appropriately fast, in addition to the boundedness assumption for some moments.

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