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ASYMPTOTIC PROPERTIES OF STATIONARY DISTRIBUTIONS IN TWO-STAGE TANDEM QUEUEING SYSTEMS

Kou Fujimoto Yukio Takahashi Naoki Makimoto
Tokyo Institute of Technology

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Abstract This paper is concerned with geometric decay properties of the joint queue length distribution $p(n_1, n_2)$ in two-stage tandem queueing system $PH/PH/c_1 \rightarrow /PH/c_2$. We prove that, under some conditions, $p(n_1, n_2) \sim C(n_2)\eta^{n_1}$ as $n_1 \rightarrow \infty$ and $p(n_1, n_2) \sim \overline{C}(n_1)\overline{\eta}^{n_2}$ as $n_2 \rightarrow \infty$. We also obtain the asymptotic form of state probabilities when n_1 is large or when n_2 is large. These results prove a part of the conjecture of a previous paper [1]. The proof is a direct application of a theorem in [7] which proves geometric decay property of the stationary distribution in a quasi-birth-and-death process with a countable number of phases in each level.

1. Introduction

Tandem queueing systems are basic models in the theory of queues and have been studied for a long time. However, because of complexities of their stochastic structure, their properties are scarcely known except for cases with product form solutions. They are simplest models of queueing networks as well as direct extensions of single queueing systems. Hence the study of them are expected to connect the theory of single queueing systems with that of queueing networks. In this paper, we prove geometric decay of the stationary state probability in a two-stage multi-server tandem queueing system $PH/PH/c_1 \rightarrow /PH/c_2$ with a buffer of infinite capacity and with heterogeneous servers.

In the ordinary one-stage queue $PH/PH/c$ with traffic intensity $\rho < 1$, it is known that the stationary distribution has a geometric tail [6]. Let $\pi(n; i_0, i_1)$ be the stationary probability that there exist n customers in the system while the phases of the arrival and service processes are i_0 and i_1 respectively, then

$$\pi(n; i_0, i_1) \sim G\, C_0(i_0)\, C_1(i_1)\, \eta^n, \qquad n \rightarrow \infty, \tag{1.1}$$

where G, $C_0(i_0)$, $C_1(i_1)$ and η are some constants and $\sim$ indicates that the ratio of both sides tends to 1. These constants other than G can be easily obtained from the phase type representations of the interarrival and service time distributions. This kind of geometric decay property is very useful, for example, on the computation of the stationary state probabilities, or on the discussion of tail probabilities for estimating very small loss probabilities (e.g. less than 10^{-9}) of the corresponding finite queue. The above result was further extended for the $GI/PH/c$ queue with heterogeneous service distributions [5].

Our main concern here is to prove a similar geometric tail property in the two-stage tandem queueing system $PH/PH/c_1 \rightarrow /PH/c_2$ with heterogeneous servers.

In a previous paper [1], the authors have made a conjecture on the geometric decay of the stationary state probability in a single-server two-stage tandem queueing system $PH/PH/1 \rightarrow /PH/1$ through an extensive numerical experiment. Let $\pi(n_1, n_2; i_0, i_1, i_2)$ be the stationary probability that there exist n_1 customers in the fist stage and n_2 customers in the second stage while the phases of the arrival process and the two service processes are i_0, i_1 and i_2, respectively. Then the conjecture asserts that the stationary state probability decays geometrically as n_1 and/or n_2 become large but decay rates and multiplicative constants may be different according to the ratio of n_1 and n_2:

$$\pi(n_1, n_2; i_0, i_1, i_2) \sim \begin{cases} G\ C_0(i_0)\ C_1(i_1)\ C_2(i_2)\ \eta_1^{n_1}\eta_2^{n_2}, \\ \qquad\qquad \text{for large } n_1 \text{ and/or } n_2 \text{ such that } \ n_2 < \alpha\, n_1, \\ \overline{G}\ \overline{C}_0(i_0)\ \overline{C}_1(i_1)\ \overline{C}_2(i_2)\ \overline{\eta}_1^{n_1}\overline{\eta}_2^{n_2}, \\ \qquad\qquad \text{for large } n_1 \text{ and/or } n_2 \text{ such that } \ n_1 < \alpha^{-1}\, n_2, \end{cases} \tag{1.2}$$

where $\alpha = -\ln(\eta_1/\overline{\eta}_1)/\ln(\eta_2/\overline{\eta}_2)$ and constants η_k, $\overline{\eta}_k$ $(k = 1, 2)$ and $C_k(i_k)$, $\overline{C}_k(i_k)$ $(k = 0, 1, 2)$ are determined from the phase-type representations of the interarrival and service time distributions.

In this paper, under a certain condition, we prove the geometric decay property (1.2) for the multi-server case $PH/PH/c_1 \to /PH/c_2$ in two special cases, the case where $n_1 \to \infty$ with n_2 being fixed and the case where $n_2 \to \infty$ with n_1 being fixed. The proof uses a result on the Matrix-geometric form solution of a quasi-birth-and-death (QBD) process with a *countable* number of phases in each level [7].

The remainder of the paper is constructed as follows. In Section 2, we describe our two-stage tandem queueing model and state our main theorems in Section 3. The result of [7] is briefly summarized in Section 4, and we prove the main theorems for a single-server case $PH/PH/1 \to /PH/1$ in Sections 5 and 6. In Section 7 we give an outline of the proof for the multi-server case $PH/PH/c_1 \to /PH/c_2$. In many places of the proofs, we use properties of solutions of four key systems of equations given in Section 3. These properties are proved in Appendix.

2. Model Description

We denote by $PH(\boldsymbol{a}, \boldsymbol{\Phi})$ a phase-type distribution represented by a continuous-time, finite-state, absorbing Markov chain with initial probability vector $\tilde{\boldsymbol{a}} = (\boldsymbol{a}, 0)$ and transition rate matrix $\tilde{\boldsymbol{\Phi}} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\gamma} \\ \boldsymbol{0} & 0 \end{bmatrix}$ (see [4]). The phase-type distribution is said to be *irreducible* if $\boldsymbol{\gamma}\boldsymbol{a} + \boldsymbol{\Phi}$ is irreducible, or equivalently $-\boldsymbol{a}\boldsymbol{\Phi}^{-1} > \boldsymbol{0}$.

We consider a two-stage tandem queueing system (Figure 1). Customers arrive at the first stage to be served there, move to the second to be served there again, and then go out of the system. The k-th stage $(k = 1, 2)$ has c_k servers and a buffer of infinite capacity, so that neither loss nor blocking occurs. Interarrival times of customers are independent and identically distributed (i.i.d.) random variables subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\alpha}, \boldsymbol{T})$. Service times at the j-th server of the k-th stage $(j = 1, 2, \ldots, c_k)$ are also i.i.d. variables subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\beta}_{kj}, \boldsymbol{S}_{kj})$. These interarrival and service times are assumed to be mutually independent. Customers are served under the first-come-first-served (FCFS) discipline and those who find multiple servers being idle choose an idle server randomly according to state-dependent probabilities.

The state of the system is represented by a vector $(n_1, n_2; i_0; i_{11}, \ldots, i_{1c_1}; i_{21}, \ldots, i_{2c_2})$, where n_k is the number of customers in the k-th stage, i_0 is the phase of the arrival process, and i_{kj} is the phase of the service process at the j-th server of the k-th stage $(j = 1, 2, \ldots, c_k;\ k = 1, 2)$. The index i_{kj} is interpreted to be equal to zero if the correspond-

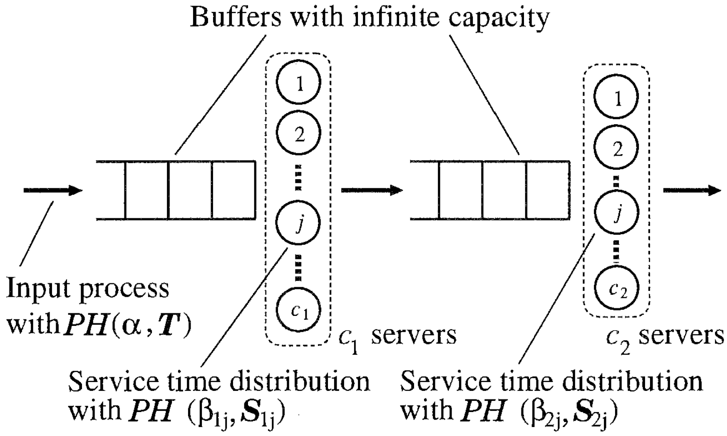

Figure 1: Two-stage tandem queueing system

ing server is idle. Then the system behaves as a continuous-time Markov chain, which we denote by $\{X(t)\}$. For brevity of notation, we sometimes abbreviate the vector representation $(n_1, n_2; i_0; i_{11}, \ldots, i_{1c_1}; i_{21}, \ldots, i_{2c_2})$ as $(n_1, n_2; i_0, i_1, i_2)$, where i_1 should be interpreted as a vector $(i_{11}, \ldots, i_{1c_1})$ and i_2 as a vector $(i_{21}, \ldots, i_{2c_2})$.

We denote the traffic intensity at the k-th stage by $\rho_k = \lambda / \sum_{j=1}^{c_k} \mu_{kj}$ where $1/\lambda$ is the mean interarrival time and $1/\mu_{kj}$ is the mean service time at the j-th server of the k-th stage, and assume $\rho_1, \rho_2 < 1$ so that the chain is stable and has stationary state probabilities $\pi(n_1, n_2; i_0; i_{11}, \ldots, i_{1c_1}; i_{21}, \ldots, i_{2c_2})$.

3. Main Theorems

The (marginal) queue-length distribution of the first stage clearly has a geometric tail as proved in [6], since the behavior of the first stage is not affected by that of the second stage. Our concern is the tail property of the joint queue-length distribution of the first and the second stages or the asymptotic behavior of the stationary state probabilities.

We prepare some notations. Hereafter, k represents the stage number and takes a value 1 or 2, and j represents the server number and runs from 1 to c_k. We denote by $\boldsymbol{I}$ the identity matrix and $\boldsymbol{e}$ the column vector of all entries equal to 1. The order of them may be finite or infinite and is understood so that expressions are well defined. If we need to emphasize the order of them, we attach a suffix "0" or a double suffix "kj". For example, $\boldsymbol{I}_0$ is the identity matrix of the same order as $\boldsymbol{T}$ and $\boldsymbol{e}_{kj}$ is the column vector of the same order as $\boldsymbol{S}_{kj}$ with all entries equal to 1.

We set

$$\boldsymbol{\gamma}_0 = -\boldsymbol{T}\boldsymbol{e}_0 \qquad \text{and} \qquad \boldsymbol{\gamma}_{kj} = -\boldsymbol{S}_{kj}\,\boldsymbol{e}_{kj}.$$

Then the Laplace-Stieltjes Transforms of the interarrival and service time distributions are given by

$$T^*(s) = \boldsymbol{\alpha}\,(s\boldsymbol{I}_0 - \boldsymbol{T})^{-1}\boldsymbol{\gamma}_0 \qquad \text{and} \qquad S^*_{kj}(s) = \boldsymbol{\beta}_{kj}\,(s\boldsymbol{I}_{kj} - \boldsymbol{S}_{kj})^{-1}\boldsymbol{\gamma}_{kj}\,. \tag{3.1}$$

Now we shall state our main results. We start from the case with $n_1 \to \infty$.

Consider the system of equations for h, s_0, s_{11}, $\ldots$, s_{1c_1}

$$\begin{cases} T^*(s_0) = h, \\ S^*_{1j}(s_{1j}) = h^{-1}, \qquad j = 1, 2, \ldots, c_1, \\ s_0 + s_{11} + \cdots + s_{1c_1} = 0. \end{cases} \tag{3.2}$$

As will be proved in Lemma 8.1 in Appendix, the system of equations (3.2) has two solutions, one of which is $(h, s_0, s_{11}, \ldots, s_{1c_1}) = (1, 0, 0, \ldots, 0)$. We will denote the other solution as $(h, s_0, s_{11}, \ldots, s_{1c_1}) = (\eta_1, \sigma_0, \sigma_{11}, \ldots, \sigma_{1c_1})$. Then from the stability condition $\rho_1 < 1$, we will see in the same lemma that $0 < \eta_1 < 1$, $\sigma_0 > 0$ and $\sigma_{1j} < 0$.

Using η_1 above we consider another system of equations for $h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}$

$$\begin{cases} T^*(s_0) = \eta_1, \\ S^*_{1j}(s_{1j}) = \eta_1^{-1} h, \qquad j = 1, 2, \ldots, c_1, \\ S^*_{2j}(s_{2j}) = h^{-1}, \qquad j = 1, 2, \ldots, c_2, \\ s_0 + s_{11} + \cdots + s_{1c_1} + s_{21} + \cdots + s_{2c_2} = 0. \end{cases} \tag{3.3}$$

The system of equations (3.3) has again two solutions as proved in Lemma 8.2. One of them is $(h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}) = (1, \sigma_0, \sigma_{11}, \ldots, \sigma_{1c_1}, 0, \ldots, 0)$, and we will denote the other as $(h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}) = (\eta_2, \omega_0, \omega_{11}, \ldots, \omega_{1c_1}, \omega_{21}, \ldots, \omega_{2c_2})$. Clearly $\omega_0 = \sigma_0$. Associated with this solution, we introduce row vectors

$$\boldsymbol{w}_0 = \boldsymbol{\alpha} \left(\omega_0 \boldsymbol{I}_0 - \boldsymbol{T}\right)^{-1} \qquad \text{and} \qquad \boldsymbol{w}_{kj} = \boldsymbol{\beta}_{kj} \left(\omega_{kj} \boldsymbol{I}_{kj} - \boldsymbol{S}_{kj}\right)^{-1}. \tag{3.4}$$

Using η_1, η_2 and vectors above, the first theorem is stated as follows.

Theorem 3.1. If $\eta_2 < 1$, for fixed n_2, i_0, $i_1 = (i_{11}, \ldots, i_{1c_1})$ and $i_2 = (i_{21}, \ldots, i_{2c_2})$, the stationary state probability decays geometrically with rate η_1 as $n_1 \to \infty$:

$$\pi(n_1, n_2; i_0, i_1, i_2) \sim G_1(n_2; i_0; i_1, i_2)\ \eta_1^{n_1}. \tag{3.5}$$

The multiplicative constant $G_1(n_2; i_0, i_1, i_2)$ decays geometrically with rate η_2 as $n_2 \to \infty$:

$$G_1(n_2; i_0, i_1, i_2) \sim G_2\, C_0(i_0)\, C_1(i_1)\, C_2(i_2)\, \eta_2^{n_2}, \tag{3.6}$$

with

$$C_1(i_1) = C_{11}(i_{11}) \cdots C_{1c_1}(i_{1c_1}) \quad \text{and} \quad C_2(i_2) = C_{21}(i_{21}) \cdots C_{2c_2}(i_{2c_2}),$$

where $C_0(i)$ is the i-th element of $\boldsymbol{w}_0$, $C_{kj}(i)$ is the i-th element of $\boldsymbol{w}_{kj}$, and G_2 is a constant independent of n_2, i_0, i_1 and i_2.

Next we shall state our result for the case where $n_2 \to \infty$. We will use symbols with bars for quantities related to this case.

As will be proved in Lemma 8.3 in Appendix, the system of equations for $h, s_0, s_{21}, \ldots, s_{2c_2}$

$$\begin{cases} T^*(s_0) = h, \\ S^*_{2j}(s_{2j}) = h^{-1}, \qquad j = 1, 2, \ldots, c_2, \\ s_0 + s_{21} + \cdots + s_{2c_2} = 0 \end{cases} \tag{3.7}$$

has two solutions, one of which is $(h, s_0, s_{21}, \ldots, s_{2c_2}) = (1, 0, 0, \ldots, 0)$. We will denote the other solution as $(h, s_0, s_{21}, \ldots, s_{2c_2}) = (\overline{\eta}_2, \overline{\sigma}_0, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{2c_2})$. For the solution, from the same lemma, we see that $0 < \overline{\eta}_2 < 1$, $\overline{\sigma}_0 > 0$ and $\overline{\sigma}_{2j} < 0$.

For $\overline{\eta}_2$ above, from Lemma 8.4, the system of equations for $h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}$

$$\begin{cases} T^*(s_0) = h, \\ S^*_{1j}(s_{1j}) = \overline{\eta}_2 h^{-1}, \qquad j = 1, 2, \ldots, c_1, \\ S^*_{2j}(s_{2j}) = \overline{\eta}_2^{-1}, \qquad j = 1, 2, \ldots, c_2, \\ s_0 + s_{11} + \cdots + s_{1c_1} + s_{21} + \cdots + s_{2c_2} = 0. \end{cases} \tag{3.8}$$

has also two solutions, one of which is $(h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}) = (1, \overline{\sigma}_0, 0, \ldots, 0, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{2c_2})$. The other solution is denoted as $(h, s_0, s_{11}, \ldots, s_{1c_1}, s_{21}, \ldots, s_{2c_2}) = (\overline{\eta}_1, \overline{\omega}_0, \overline{\omega}_{11}, \ldots, \overline{\omega}_{1c_1}, \overline{\omega}_{21}, \ldots, \overline{\omega}_{2c_2})$. Clearly $\overline{\omega}_{2j} = \overline{\sigma}_{2j}$. Associated with this solution, we introduce row vectors

$$\overline{\boldsymbol{w}}_0 = \boldsymbol{\alpha}\,(\overline{\omega}_0 \boldsymbol{I}_0 - \boldsymbol{T})^{-1} \quad \text{and} \quad \overline{\boldsymbol{w}}_{kj} = \boldsymbol{\beta}_{kj}\,(\overline{\omega}_{kj} \boldsymbol{I}_{kj} - \boldsymbol{S}_{kj})^{-1}. \tag{3.9}$$

Using η_1, $\overline{\eta}_2$, $\overline{\eta}_1$ and vectors above, the second theorem is stated as follows.

Theorem 3.2. If $\eta_1 < \overline{\eta}_2$ and $\overline{\eta}_1 < \overline{\eta}_2$, for fixed n_1, i_0, $i_1 = (i_{11}, \ldots, i_{1c_1})$ and $i_2 = (i_{21}, \ldots, i_{2c_2})$, the stationary state probability decays geometrically with rate $\overline{\eta}_2$ as $n_2 \to \infty$:

$$\pi(n_1, n_2; i_0, i_1, i_2) \sim \overline{G}_2(n_1; i_0; i_1, i_2)\ \overline{\eta}_2^{n_2}. \tag{3.10}$$

The multiplicative constant $\overline{G}_2(n_1; i_0, i_1, i_2)$ decays geometrically with rate $\overline{\eta}_1$ as $n_1 \to \infty$:

$$\overline{G}_2(n_1; i_0, i_1, i_2) \sim \overline{G}_1\, \overline{C}_0(i_0)\, \overline{C}_1(i_1)\, \overline{C}_2(i_2)\, \overline{\eta}_1^{n_1} \tag{3.11}$$

with

$$\overline{C}_1(i_1) = \overline{C}_{11}(i_{11}) \cdots \overline{C}_{1c_1}(i_{1c_1}) \qquad \text{and} \qquad \overline{C}_2(i_2) = \overline{C}_{21}(i_{21}) \cdots \overline{C}_{2c_2}(i_{2c_2}),$$

where $\overline{C}_0(i)$ is the i-th element of $\overline{\boldsymbol{w}}_0$, $\overline{C}_{kj}(i)$ is the i-th element of $\overline{\boldsymbol{w}}_{kj}$, and $\overline{G}_1$ is a constant independent of n_1, i_0, i_1 and i_2.

Remark 1. The decay rates η_k and $\overline{\eta}_k$ have the following properties. These properties are easily proved from lemmas in Appendix.

1. The decay rate η_1 is a monotone increasing function of ρ_1, and $\eta_1 \downarrow 0$ as $\rho_1 \downarrow 0$ while $\eta_1 \uparrow 1$ as $\rho_1 \uparrow 1$. The other decay rate η_2 is less than 1 if ρ_2 is small but it may exceed 1 if ρ_2 becomes large. η_2 can be regarded as a function of both ρ_1 and ρ_2. For fixed ρ_1, it is a monotone increasing function of ρ_2 and $\eta_2 \downarrow 0$ as $\rho_2 \downarrow 0$.
2. A numerical experiment shows that $\eta_2 < 1$ in most of two-stage tandem queueing systems, and hence Theorem 3.1 holds in a wide range.
3. The decay rate $\overline{\eta}_2$ is a monotone increasing function of ρ_2, and $\overline{\eta}_2 \downarrow 0$ as $\rho_2 \downarrow 0$ while $\overline{\eta}_2 \uparrow 1$ as $\rho_2 \uparrow 1$. The other decay rate $\overline{\eta}_1$ is a function of ρ_1 and ρ_2, and for fixed ρ_2 it is a monotone increasing function of ρ_1. As $\rho_1 \downarrow 0$, $\overline{\eta}_1 \downarrow 0$, and as $\rho_1 \uparrow 1$, $\overline{\eta}_1 \uparrow \eta_2$. This means that $\overline{\eta}_1$ may exceed 1.
4. For the condition $\eta_1 < \overline{\eta}_2$ to hold, ρ_2 must be greater than some positive value. Hence Theorem 3.2 holds only in some limited cases.

Remark 2. The authors have never succeeded to give an intuitive interpretation for the condition $\eta_2 < 1$ of Theorem 3.1 and the condition $\eta_1 < \overline{\eta}_2$ and $\overline{\eta}_1 < \overline{\eta}_2$ of Theorem 3.2. Related discussions are given for single-server tandem queues $MAP/PH/1 \to /PH/1$ in [3] and $GI/M/1 \to /M/1$ in [2].

Remark 3. It is well known that the marginal queue-length distribution of the first stage

$$p_1(n_1) = \sum_{n_2, i_0, i_1, i_2} \pi(n_1, n_2; i_0, i_1, i_2)$$

has geometric tail with decay rate η_1. It also can be shown that, if the conditions in

Theorem 3.2 hold, the marginal queue-length distribution of the second stage

$$p_2(n_2) = \sum_{n_1, i_0, i_1, i_2} \pi(n_1, n_2; i_0, i_1, i_2)$$

has geometric tail, too, but with decay rate $\overline{\eta}_2$. Its proof requires some extra pages, and it will be presented elsewhere.

Remark 4. The constant $G_1(n_2; i_0, i_1, i_2)$ in Theorem 3.1 is given by the corresponding element of $\boldsymbol{p}$ in Lemma 5.2 up to a multiplicative constant. Using the geometric decay property of $p_1(n_1)$ in Remark 3, we can see that $\boldsymbol{p}$ up to a multiplicative constant gives the conditional state probabilities when n_1 is sufficiently large. From the proof of the lemma, $\boldsymbol{p}$ is directly derived from stationary distribution $\boldsymbol{p}_D$ of a certain QBD process with a finite number of phases. Since $\boldsymbol{p}_D$ is of the matrix-geometric form, we can easily obtain the value of $G_1(n_2; i_0, i_1, i_2)$ by numerical computation.

The constant $G_2(n_1; i_0, i_1, i_2)$ in Theorem 3.2 corresponds to the element of $\overline{\boldsymbol{p}}$ given in Lemma 6.2. It is also easy to get numerical value of $\overline{\boldsymbol{p}}$ from the steady-state distribution of a QBD process. When n_2 is sufficiently large, using the geometric decay property of $p_2(n_2)$ in Remark 3, the conditional state probabilities is given by $\overline{\boldsymbol{p}}$ up to multiplicative constant.

4. Geometric Decay Property in a Quasi-Birth-and-Death Process with a Countable Number of Phases

To prove the theorems, we use the corollaries in [7]. These corollaries are summarized as Proposition 1 below.

Consider a continuous time positive recurrent Markov chain $\{X(t)\}$ on the state space $\mathcal{S} = \{(m, i);\, m, i = 0, 1, 2, \ldots\}$. The state space $\mathcal{S}$ is partitioned into subsets $\mathcal{L}_m = \{(m, i); i = 0, 1, 2, \ldots\}$, $m = 0, 1, 2, \ldots$, called *levels*. When partitioned by levels, the transition rate matrix $\boldsymbol{Q}$ of $\{X(t)\}$ is assumed to have a block-tridiagonal form:

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_1 & \boldsymbol{B} & \boldsymbol{A} & & \\ & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} & \\ & & \boldsymbol{C} & \boldsymbol{B} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{4.1}$$

Such a chain is called a *quasi-birth-and-death* (QBD) process with a countable number of phases in each level. The stationary vector $\boldsymbol{\pi}$ of the QBD process is also partitioned as $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)$ according to $\mathcal{L}_m$'s.

Proposition 1. We assume that diagonal elements of $-\boldsymbol{B}$ and $-\boldsymbol{B}_0$ are bounded by $d(< \infty)$ from above and that there exist a positive constant $\eta < 1$ and positive vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying

$$\boldsymbol{p}\left(\eta^{-1}\boldsymbol{A} + \boldsymbol{B} + \eta\boldsymbol{C}\right) = \boldsymbol{0}, \tag{4.2}$$

$$\left(\eta^{-1}\boldsymbol{A} + \boldsymbol{B} + \eta\boldsymbol{C}\right)\boldsymbol{q} = \boldsymbol{0}, \tag{4.3}$$

$$\eta^{-1}\boldsymbol{p}\,\boldsymbol{A}\,\boldsymbol{q} \neq \eta\,\boldsymbol{p}\,\boldsymbol{C}\,\boldsymbol{q}, \tag{4.4}$$

$$\boldsymbol{p}\,\boldsymbol{e} < \infty, \quad \text{and} \quad \boldsymbol{p}\,\boldsymbol{q} < \infty. \tag{4.5}$$

If $\boldsymbol{\pi}_1 \boldsymbol{q} < \infty$ and if the matrix $\boldsymbol{Q}'$ formed from $\boldsymbol{Q}$ by deleting rows and columns corresponding to states in $\mathcal{L}_0$ is irreducible, then $\boldsymbol{\pi}$ has a geometric tail:

$$\boldsymbol{\pi}_m \sim C\, \eta^m \boldsymbol{p}, \qquad \text{as } m \to \infty. \tag{4.6}$$

For the proofs of Theorems 3.1 and 3.2 in the preceding section, we apply this proposition for partitions by the number of customers in the first or the second stage.

We also prepare some properties related to phase-type distributions. We will denote by $\otimes$ and $\oplus$ the Kronecker product and the Kronecker sum operations.

Let $PH(\boldsymbol{a}, \boldsymbol{\Phi})$ be an irreducible phase-type distribution and put $\boldsymbol{\gamma} = -\boldsymbol{\Phi}\boldsymbol{e}$. Then the LST of the distribution and its derivative are given by

$$\Phi^*(s) = \boldsymbol{a}(s\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\gamma} \qquad \text{and} \qquad \Phi^{*'}(s) = -\boldsymbol{a}(s\boldsymbol{I} - \boldsymbol{\Phi})^{-2}\boldsymbol{\gamma}. \tag{4.7}$$

Especially, the mean of the distribution is given by $-\Phi^{*'}(0) = \boldsymbol{a}(-\boldsymbol{\Phi})^{-2}\boldsymbol{\gamma} = \boldsymbol{a}(-\boldsymbol{\Phi})^{-1}\boldsymbol{e}$. Note that $\Phi^*(s)$ can be considered as a function of s on the interval (ϕ, ∞) where $\phi < 0$ is the abscissa of convergence.

Lemma 4.1. For $h > 0$ and $s > \phi$, the equation

$$\boldsymbol{x}(h^{-1}\,\boldsymbol{\gamma}\,\boldsymbol{a} + \boldsymbol{\Phi}) = s\,\boldsymbol{x} \tag{4.8}$$

has a positive solution $\boldsymbol{x}$ if and only if $\Phi^*(s) = h$. If $\Phi^*(s) = h$, then $\boldsymbol{x} = \boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}$ is a unique positive solution of (4.8) up to a multiplicative constant.

Proof. From (4.8) we have $\boldsymbol{x}\,(s\boldsymbol{I} - \boldsymbol{\Phi}) = (h^{-1}\boldsymbol{x}\boldsymbol{\gamma})\,\boldsymbol{a}$ and hence

$$\boldsymbol{x} = (h^{-1}\boldsymbol{x}\boldsymbol{\gamma})\,\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}. \tag{4.9}$$

Postmultiplying $\boldsymbol{\gamma}$, we have $\boldsymbol{x}\boldsymbol{\gamma} = h^{-1}\boldsymbol{x}\boldsymbol{\gamma}\,\Phi^*(s)$. This implies that, if the equation (4.8) has a positive solution, then $\Phi^*(s) = h$. Conversely, if $\Phi^*(s) = h$, by substituting $\boldsymbol{x}$ with $\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}$ the left hand side of (4.8) is rewritten as

$$\begin{aligned}
\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}(h^{-1}\,\boldsymbol{\gamma}\,\boldsymbol{a} + \boldsymbol{\Phi}) &= h^{-1}\Phi^*(s)\,\boldsymbol{a} + \boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\Phi} \\
&= \boldsymbol{a} - \boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}(s\,\boldsymbol{I} - \boldsymbol{\Phi}) + s\,\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \\
&= s\,\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}.
\end{aligned}$$

Hence $\boldsymbol{x} = \boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}$ is a solution of (4.8). The positivity of $\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}$ is clear from the irreducibility of the distribution. The equation (4.9) indicates that $\boldsymbol{a}\,(s\,\boldsymbol{I} - \boldsymbol{\Phi})^{-1}$ is a unique solution up to a multiplicative constant. ∎

Lemma 4.2. For irreducible phase-type distributions $PH(\boldsymbol{a}_i, \boldsymbol{\Phi}_i)$ with $\gamma_i = -\boldsymbol{\Phi}_i\boldsymbol{e}$, $i = 1, 2, \ldots, c$, and for positive numbers $h_1, h_2, \ldots, h_c$, the vector equation

$$\boldsymbol{x}\left\{\left(h_1^{-1}\,\boldsymbol{\gamma}_1\,\boldsymbol{a}_1 + \boldsymbol{\Phi}_1\right) \oplus \cdots \oplus \left(h_c^{-1}\,\boldsymbol{\gamma}_c\,\boldsymbol{a}_c + \boldsymbol{\Phi}_c\right)\right\} = \boldsymbol{0} \tag{4.10}$$

has a positive solution $\boldsymbol{x}$ if and only if there exit real numbers $s_1, s_2, \ldots, s_c$ such that $\Phi_i^*(s_i) = h_i$, $i = 1, 2, \ldots, c$, and $s_1 + s_2 + \cdots + s_c = 0$. The solution is given by $\boldsymbol{x} = \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_c$ with $\boldsymbol{x}_i = \boldsymbol{a}_i\,(s_i\,\boldsymbol{I}_i - \boldsymbol{\Phi}_i)^{-1}$, $i = 1, 2, \ldots, c$, up to a multiplicative constant.

Proof. First we prove "if" part. From Lemma 4.1 we have

$$\begin{aligned}
&\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_c \left\{ \left(h_1^{-1} \gamma_1 \boldsymbol{a}_1 + \boldsymbol{\Phi}_1 \right) \oplus \cdots \oplus \left(h_c^{-1} \gamma_c \boldsymbol{a}_c + \boldsymbol{\Phi}_c \right) \right\} \\
&= \boldsymbol{x}_1 \left(h_1^{-1} \gamma_1 \boldsymbol{a}_1 + \boldsymbol{\Phi}_1 \right) \otimes \boldsymbol{x}_2 \otimes \cdots \otimes \boldsymbol{x}_c + \cdots + \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_{c-1} \otimes \boldsymbol{x}_c \left(h_c^{-1} \gamma_c \boldsymbol{a}_c + \boldsymbol{\Phi}_c \right) \\
&= (s_1 + \cdots + s_c)\, \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_c \;=\; \mathbf{0}.
\end{aligned}$$

Thus $\boldsymbol{x}$ is a solution of (4.10). For the proof of "only if" part, we write $\boldsymbol{\Psi} = \left(h_1^{-1} \gamma_1 \boldsymbol{a}_1 + \boldsymbol{\Phi}_1 \right) \oplus \cdots \oplus \left(h_c^{-1} \gamma_c \boldsymbol{a}_c + \boldsymbol{\Phi}_c \right)$. From the irreducibility of $PH(\boldsymbol{a}_i, \boldsymbol{\Phi}_i)$, the matrix $\boldsymbol{\Psi}$ is irreducible and its Perron-Frobenius eigenvalue is simple. The equation (4.10) implies the Perron-Frobenius eigenvalue is equal to zero.

Since $\boldsymbol{\Psi}$ is a Kronecker sum of c smaller matrices, its simple Perron-Frobenius eigenvalue is a sum of Perron-Frobenius eigenvalues of the smaller matrices. From Lemma 4.1 such eigenvalues have to satisfy $\Phi_i^*(s_i) = h_i$ and $s_1 + s_2 + \cdots + s_c = 0$. ∎

5. Proof of Theorem 3.1 for Single Server Case

First we prove Theorem 3.1 for the single server system $PH/PH/1 \to /PH/1$. In this case, $c_1 = c_2 = 1$ and the state of the Markov chain $\{X(t)\}$ is written as $(n_1, n_2; i_0, i_1, i_2)$ in which i_1 and i_2 are no longer vectors but integers. Double suffices in symbols such as $\boldsymbol{S}_{kj}$ and ω_{kj} are also replaced with a single suffix as $\boldsymbol{S}_k$ and ω_k.

We arrange the states $(n_1, n_2; i_0, i_1, i_2)$ in lexicographic order and partition the state space according to n_1, i.e., we let

$$\mathcal{L}_m = \{(n_1, n_2; i_0, i_1, i_2) \mid n_1 = m\}, \qquad m = 0, 1, 2, \ldots. \tag{5.1}$$

We denote by $\boldsymbol{Q}$ the transition rate matrix of the chain corresponding to the arrangement above and by $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)$ the stationary vector partitioned according to $\mathcal{L}_m$'s. Then $\boldsymbol{Q}$ is of the block-tridiagonal form as (4.1) with

$$\boldsymbol{A} = \begin{pmatrix} \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 & & & & \\ & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 & & & \\ & & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 & & \\ & & & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 & \\ & & & & \ddots \end{pmatrix}, \tag{5.2}$$

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{T} \oplus \boldsymbol{S}_1 & & & & \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & & & \\ & \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & & \\ & & \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & \\ & & & \ddots & \ddots \end{pmatrix}, \tag{5.3}$$

and

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 & & & \\ & \boldsymbol{O} & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2 & & \\ & & \boldsymbol{O} & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2 & \\ & & & \boldsymbol{O} & \ddots \\ & & & & \ddots \end{pmatrix}. \tag{5.4}$$

Then $\{X(t)\}$ is regarded as a QBD process with a countable number of phases in each level.

Now we shall apply Proposition 1 and prove Theorem 3.1 by taking $\eta = \eta_1$ and suitably defining vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. First we introduce several matrices and vectors including $\boldsymbol{q}$, and then we check in Lemmas 5.1 $\sim$ 5.3 that the condition (4.2) $\sim$ (4.5) of the proposition hold. The geometric decay of the limiting vector $\boldsymbol{p}$ will be proved in Lemma 5.4.

Let

$$\boldsymbol{K} \equiv \eta_1^{-1}\boldsymbol{A} + \boldsymbol{B} + \eta_1\boldsymbol{C} = \begin{pmatrix} \tilde{\boldsymbol{B}}_0 & \tilde{\boldsymbol{A}}_0 & & & \\ \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} & & \\ & \tilde{\boldsymbol{C}} & \tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} & \\ & & \tilde{\boldsymbol{C}} & \tilde{\boldsymbol{B}} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Then from (5.2), (5.3) and (5.4)

$$\begin{aligned} \tilde{\boldsymbol{A}}_0 &= \eta_1 \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2, & \tilde{\boldsymbol{A}} &= \eta_1 \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2, \\ \tilde{\boldsymbol{B}}_0 &= \eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} \otimes \boldsymbol{I}_1 + \boldsymbol{T} \oplus \boldsymbol{S}_1, & \tilde{\boldsymbol{B}} &= \eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 + \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2, \\ \tilde{\boldsymbol{C}}_1 &= \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2, \quad \text{and} & \tilde{\boldsymbol{C}} &= \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2. \end{aligned} \tag{5.5}$$

We define row vectors and column vectors as

$$\begin{aligned} &\boldsymbol{u}_0 = \boldsymbol{\alpha}(\sigma_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}, \qquad \boldsymbol{u}_1 = \boldsymbol{\beta}_1(\sigma_1\boldsymbol{I}_1 - \boldsymbol{S}_1)^{-1}, \qquad \boldsymbol{u}_2 = \boldsymbol{\beta}_2(-\boldsymbol{S}_2)^{-1}, \\ &\boldsymbol{v}_0 = (\sigma_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}\boldsymbol{\gamma}_0 \quad \text{and} \quad \boldsymbol{v}_1 = (\sigma_1\boldsymbol{I}_1 - \boldsymbol{S}_1)^{-1}\boldsymbol{\gamma}_1 . \end{aligned} \tag{5.6}$$

Since $\omega_0 = \sigma_0$, the vector $\boldsymbol{u}_0$ is the same as $\boldsymbol{w}_0$ defined earlier. But for symmetry of expressions, we introduce another symbol here. Note that, from Lemma 4.1, $\boldsymbol{u}_0$ is a solution of the equation $\boldsymbol{u}_0(\eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} + \boldsymbol{T}) = \sigma_0\boldsymbol{u}_0$. Other vectors are solutions of similar equations, too. These equations will be frequently used in the proofs of subsequent lemmas.

From (4.7) and the fact that $(\eta_1, \sigma_0, \sigma_1)$ is a solution of the equation (3.2), these vectors are related with LST of interarrival and service time distributions as follows.

$$\begin{aligned} &\boldsymbol{u}_0\boldsymbol{\gamma}_0 = \boldsymbol{\alpha}\boldsymbol{v}_0 = T^*(\sigma_0) = \eta_1, \qquad \boldsymbol{u}_1\boldsymbol{\gamma}_1 = \boldsymbol{\beta}_1\boldsymbol{v}_1 = S_1^*(\sigma_1) = \eta_1^{-1}, \\ &\boldsymbol{u}_0\boldsymbol{v}_0 = -T^{*\prime}(\sigma_0), \qquad \boldsymbol{u}_1\boldsymbol{v}_1 = -S_1^{*\prime}(\sigma_1) \quad \text{and} \quad \boldsymbol{u}_2\boldsymbol{e}_2 = -S_2^{*\prime}(0) . \end{aligned} \tag{5.7}$$

Since $\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}} = (\eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} + \boldsymbol{T}) \oplus (\eta_1\boldsymbol{\gamma}_1\boldsymbol{\beta}_1 + \boldsymbol{S}_1) \oplus (\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 + \boldsymbol{S}_2)$, it follows from Lemma 4.2 that

$$(\boldsymbol{u}_0 \otimes \boldsymbol{u}_1 \otimes \boldsymbol{u}_2)(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}) = \boldsymbol{0}, \qquad (\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}})(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) = \boldsymbol{0}. \tag{5.8}$$

Let us define a column vector

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2 \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2 \\ \vdots \end{pmatrix}. \tag{5.9}$$

Lemma 5.1. The vector $\boldsymbol{q}$ is positive and satisfies $\boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}$.

Proof. The positivity of $\boldsymbol{q}$ is clear from the definition. The vector $\boldsymbol{Kq}$ is written as

$$\begin{pmatrix} \tilde{\boldsymbol{A}}_0(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) + \tilde{\boldsymbol{B}}_0(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1) \\ (\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}})(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) + \tilde{\boldsymbol{C}}_1(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1) \\ (\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}})(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) \\ (\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}})(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) \\ \vdots \end{pmatrix}.$$

From (5.8) the subvectors of $\boldsymbol{Kq}$ corresponding to $\mathcal{L}_m$, $(m \geq 2)$, are equal to zero vectors. For the first two subvectors we can also check that they are equal to zero vectors. ∎

To prove the next lemma, we need another notation. For an arbitrary column vector $\boldsymbol{\phi}$ we let $diag(\boldsymbol{\phi})$ be the diagonal matrix having elements of $\boldsymbol{\phi}$ in its diagonals. The operator $diag(\cdot)$ satisfies the following relations for column vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$:

$$diag(\boldsymbol{\phi} \otimes \boldsymbol{\psi}) = diag(\boldsymbol{\phi}) \otimes diag(\boldsymbol{\psi}), \qquad diag(\boldsymbol{\phi} \otimes \boldsymbol{\psi})^{-1} = diag(\boldsymbol{\phi})^{-1} \otimes diag(\boldsymbol{\psi})^{-1}.$$

Lemma 5.2. If $\eta_2 < 1$, there exists a positive vector $\boldsymbol{p}$ such that

$$\boldsymbol{pK} = \boldsymbol{0}, \qquad \boldsymbol{pe} < \infty, \quad \text{and} \quad \boldsymbol{pq} < \infty.$$

Proof. To use a property of an ergodic Markov chain, we transform $\boldsymbol{K}$ so that it becomes a transition rate matrix. Let $\boldsymbol{D} = diag(\boldsymbol{q})$ and

$$\boldsymbol{K}_D \equiv \boldsymbol{D}^{-1}\boldsymbol{KD} = \begin{pmatrix} \tilde{\boldsymbol{D}}_0^{-1}\tilde{\boldsymbol{B}}_0\tilde{\boldsymbol{D}}_0 & \tilde{\boldsymbol{D}}_0^{-1}\tilde{\boldsymbol{A}}_0\tilde{\boldsymbol{D}} & & & \\ \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}_1\tilde{\boldsymbol{D}}_0 & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{D}} & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}} & & \\ & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}} & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{D}} & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}} & \\ & & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}} & \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{D}} & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $\tilde{\boldsymbol{D}}_0 = diag(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1)$ and $\tilde{\boldsymbol{D}} = diag(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2)$. Since $\boldsymbol{K}_D\boldsymbol{e} = \boldsymbol{D}^{-1}\boldsymbol{Kq} = \boldsymbol{0}$, $\boldsymbol{K}_D$ is a transition rate matrix of a QBD process with a *finite* number of phases in each level. Theorem 3.1.1 in [4] says that $\boldsymbol{K}_D$ is ergodic if and only if

$$\tilde{\boldsymbol{\pi}}(\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}})\boldsymbol{e} < \tilde{\boldsymbol{\pi}}(\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}})\boldsymbol{e}, \tag{5.10}$$

where $\tilde{\boldsymbol{\pi}}$ is a positive row vector such that

$$\tilde{\boldsymbol{\pi}}\left(\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}}\right) = \tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{D}}^{-1}(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}})\tilde{\boldsymbol{D}} = \boldsymbol{0}.$$

From (5.8) we know that $\tilde{\boldsymbol{\pi}}$ is written as $(\boldsymbol{u}_0 \otimes \boldsymbol{u}_1 \otimes \boldsymbol{u}_2)\,\tilde{\boldsymbol{D}}$ up to a multiplicative constant. We shall evaluate the both sides of (5.10) using Lemma 4.1:

$$\begin{aligned} \tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}}\boldsymbol{e} &= (\boldsymbol{u}_0 \otimes \boldsymbol{u}_1 \otimes \boldsymbol{u}_2) \cdot (\eta_1 \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2) \cdot (\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) \\ &= \eta_1 \cdot \boldsymbol{u}_0\boldsymbol{v}_0 \cdot \eta_1^{-2} \cdot \boldsymbol{u}_2\boldsymbol{e}_2 \;=\; \eta_1^{-1}\, T^{*\prime}(\sigma_0)\, S_2^{*\prime}(0), \end{aligned}$$

and

$$\begin{aligned}\tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}}\boldsymbol{e} &= (\boldsymbol{u}_0 \otimes \boldsymbol{u}_1 \otimes \boldsymbol{u}_2) \cdot (\boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2) \cdot (\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{e}_2) \\ &= \boldsymbol{u}_0\boldsymbol{v}_0 \cdot \boldsymbol{u}_1\boldsymbol{v}_1 \cdot 1 \;=\; T^{*'}(\sigma_0)\, S_1^{*'}(\sigma_1).\end{aligned}$$

Therefore, the difference of the both sides of the inequality (5.10) is given by

$$\tilde{\boldsymbol{\pi}}(\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}})\boldsymbol{e} - \tilde{\boldsymbol{\pi}}(\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}})\boldsymbol{e} = T^{*'}(\sigma_0)\left\{\eta_1^{-1}S_2^{*'}(0) - S_1^{*'}(\sigma_1)\right\}. \tag{5.11}$$

If $\eta_2 < 1$, the quantity in the braces is positive from Lemma 8.5 in Appendix, and hence the inequality (5.10) holds. This proves that $\boldsymbol{K}_D$ is ergodic under the assumption of Theorem 3.1.

Since $\boldsymbol{K}_D$ is ergodic, there exists a positive vector $\boldsymbol{p}_D$ such that $\boldsymbol{p}_D\boldsymbol{K}_D = \boldsymbol{0}$ and $\boldsymbol{p}_D\boldsymbol{e} = 1$. Then,

$$\boldsymbol{p} = \boldsymbol{p}_D\boldsymbol{D}^{-1} \tag{5.12}$$

is the desired positive vector. In fact,

$$\boldsymbol{p}\,\boldsymbol{K} = \boldsymbol{p}_D\,\boldsymbol{K}_D\boldsymbol{D}^{-1} = \boldsymbol{0}, \qquad \boldsymbol{p}\,\boldsymbol{q} = \boldsymbol{p}_D\,\boldsymbol{e} = 1 < \infty,$$

and

$$\boldsymbol{p}\,\boldsymbol{e} < d_1\,\boldsymbol{p}\,\boldsymbol{q} < d_1 < \infty,$$

where d_1 is a positive number such that $\boldsymbol{e} < d_1\,\boldsymbol{q}$. ∎

Lemma 5.3. For $\boldsymbol{p}$ given in (5.12) we have $\eta_1^{-1}\boldsymbol{p}\boldsymbol{A}\,\boldsymbol{q} > \eta_1\,\boldsymbol{p}\,\boldsymbol{C}\,\boldsymbol{q}$.

Proof. Let

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{e}_2 \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{e}_2 \\ \vdots \end{pmatrix},$$

then

$$\begin{aligned}\boldsymbol{q} &= \boldsymbol{E}(\boldsymbol{v}_0 \otimes \boldsymbol{v}_1), \qquad \boldsymbol{K}\boldsymbol{E} = \boldsymbol{E}\left(\eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} + \boldsymbol{T}\right) \oplus (\eta_1\boldsymbol{\gamma}_1\boldsymbol{\beta}_1 + \boldsymbol{S}_1), \\ \boldsymbol{A}\boldsymbol{E} &= \boldsymbol{E}\,(\boldsymbol{\gamma}_0\boldsymbol{\alpha} \otimes \boldsymbol{I}_1) \quad \text{and} \quad \boldsymbol{C}\boldsymbol{E} = \boldsymbol{E}\,(\boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1),\end{aligned} \tag{5.13}$$

and hence

$$\eta_1^{-1}\boldsymbol{A}\,\boldsymbol{q} = \boldsymbol{E}\,(\boldsymbol{\gamma}_0 \otimes \boldsymbol{v}_1) \quad \text{and} \quad \eta_1\,\boldsymbol{C}\,\boldsymbol{q} = \boldsymbol{E}\,(\boldsymbol{v}_0 \otimes \boldsymbol{\gamma}_1). \tag{5.14}$$

Note that a postmultiplication of $\boldsymbol{E}$ implies an aggregation of states into aggregated states with common states of the arrival process and the service process at the first stage while the state of the second stage is ignored. Postmultiplying $\boldsymbol{E}$ to the equality $\boldsymbol{p}\,\boldsymbol{K} = \boldsymbol{0}$ and applying (5.13), we have

$$\boldsymbol{p}\boldsymbol{E}\left\{\left(\eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} + \boldsymbol{T}\right) \oplus (\eta_1\boldsymbol{\gamma}_1\boldsymbol{\beta}_1 + \boldsymbol{S}_1)\right\} = \boldsymbol{0}.$$

From Lemma 4.2, $\boldsymbol{p}\boldsymbol{E}$ is a constant multiple of $\boldsymbol{u}_0 \otimes \boldsymbol{u}_1$. If we let the multiplicative constant be H, we have

$$\begin{aligned}\eta_1^{-1}\boldsymbol{p}\boldsymbol{A}\boldsymbol{q} - \eta_1\boldsymbol{p}\boldsymbol{C}\boldsymbol{q} &= \boldsymbol{p}\boldsymbol{E}(\boldsymbol{\gamma}_0 \otimes \boldsymbol{v}_1) - \boldsymbol{p}\boldsymbol{E}(\boldsymbol{v}_0 \otimes \boldsymbol{\gamma}_1) \\ &= H\,\{\boldsymbol{u}_0\boldsymbol{\gamma}_0 \cdot \boldsymbol{u}_1\boldsymbol{v}_1 - \boldsymbol{u}_0\boldsymbol{v}_0 \cdot \boldsymbol{u}_1\boldsymbol{\gamma}_1\} \\ &= H\left\{-T^{*}(\sigma_0)S_1^{*'}(\sigma_1) + T^{*'}(\sigma_0)S_1^{*}(\sigma_1)\right\}.\end{aligned}$$

The quantity in the braces of the right most hand side is positive from Lemma 8.6 in Appendix, and the inequality of the lemma holds. ∎

Lemmas 5.1~5.3 above show that the constant η_1 and positive vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions (4.2)~(4.5) in Proposition 1. The remaining is to show that $\boldsymbol{\pi}_1\boldsymbol{q} < \infty$ and that the matrix $\boldsymbol{Q}'$ is irreducible. The former is trivial from the fact that there exists a positive number d_2 such that $\boldsymbol{q} < d_2\boldsymbol{e}$. The latter is easily checked using the irreducibility of the interarrival and service time distributions. Thus we have proved (3.5) of Theorem 3.1.

To prove the tail property of $\boldsymbol{p}$, we introduce a subpartition of each $\mathcal{L}_m$, $m \geq 1$,

$$l_n = \{(n_1, n_2; i_0, i_1, i_2) |\, n_1 = m,\ n_2 = n\}, \qquad n = 0, 1, 2, \ldots,$$

and we divide the vector $\boldsymbol{p}$ as $(\boldsymbol{p}(0),\ \boldsymbol{p}(1),\ \ldots)$ according to this subpartition.

Lemma 5.4. The vector $\boldsymbol{p}$ given in (5.12) has a geometric tail:

$$\boldsymbol{p}(n_2) \sim G'\, \eta_2^{n_2}\, \boldsymbol{w}_0 \otimes \boldsymbol{w}_1 \otimes \boldsymbol{w}_2, \qquad \text{as } n_2 \to \infty,$$

where G' is a constant and $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ $(= \boldsymbol{w}_{11})$ and $\boldsymbol{w}_2$ $(= \boldsymbol{w}_{21})$ are vectors defined in (3.4).

Proof. By the QBD structure of the transition rate matrix $\boldsymbol{K}_D$, we can apply the matrix analytic theory by Neuts [4]. Let $\tilde{\boldsymbol{R}}_D$ be the rate matrix of $\boldsymbol{K}_D$. Then it is the minimal non-negative solution to the matrix equation

$$\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{R}}_D\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{R}}_D^2\tilde{\boldsymbol{D}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{D}} = \boldsymbol{O}. \tag{5.15}$$

Since $\boldsymbol{p} = \boldsymbol{p}_D\boldsymbol{D}^{-1}$, we know that

$$\boldsymbol{p}(n_2) = \boldsymbol{p}(1)\, \tilde{\boldsymbol{D}}\, \tilde{\boldsymbol{R}}_D^{n_2}\, \tilde{\boldsymbol{D}}^{-1} \sim G''\, \tilde{\eta}^{n_2}\, \tilde{\boldsymbol{x}}_D\, \tilde{\boldsymbol{D}}^{-1}, \qquad \text{as } n_2 \to \infty,$$

where G'' is a constant, $\tilde{\eta}$ is the Perron-Frobenius eigenvalue of $\tilde{\boldsymbol{R}}_D$ and $\tilde{\boldsymbol{x}}_D$ is a corresponding left eigenvector: $\tilde{\boldsymbol{x}}_D\, \tilde{\boldsymbol{R}}_D = \tilde{\eta}\, \tilde{\boldsymbol{x}}_D$. Then by premultiplying $\tilde{\boldsymbol{x}}_D$ to the equality (5.15), we have

$$\begin{aligned} \boldsymbol{0} &= \tilde{\boldsymbol{x}}_D\tilde{\boldsymbol{D}}^{-1}\left(\tilde{\eta}^{-1}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} + \tilde{\eta}\tilde{\boldsymbol{C}}\right) \\ &= \tilde{\boldsymbol{x}}_D\tilde{\boldsymbol{D}}^{-1}\left[\left(\eta_1^{-1}\boldsymbol{\gamma}_0\boldsymbol{\alpha} + \boldsymbol{T}\right) \oplus \left(\eta_1^{-1}\tilde{\eta}\boldsymbol{\gamma}_1\boldsymbol{\beta}_1 + \boldsymbol{S}_1\right) \oplus \left(\tilde{\eta}^{-1}\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 + \boldsymbol{S}_2\right)\right]. \end{aligned}$$

Since $\tilde{\eta} < 1$, from Lemma 4.2 and Lemma 8.1, $\tilde{\boldsymbol{x}}_D\tilde{\boldsymbol{D}}^{-1}$ is a constant multiple of $\boldsymbol{w}_0 \otimes \boldsymbol{w}_1 \otimes \boldsymbol{w}_2$:

$$\tilde{\boldsymbol{x}}_D\tilde{\boldsymbol{D}}^{-1} = G'''\, \boldsymbol{w}_0 \otimes \boldsymbol{w}_1 \otimes \boldsymbol{w}_2.$$

This proves the lemma. ∎

Lemma 5.4 proves (3.6) of Theorem 3.1, and this completes the proof of the theorem.

6. Proof of Theorem 3.2 for Single Server Case

Next we prove Theorem 3.2 for the single server system $PH/PH/1 \to /PH/1$. In this section we use the same convention for the double suffices as in the last section.

We rearrange the states $(n_1, n_2; i_0, i_1, i_2)$ of $\{X(t)\}$ first in the order of n_2 and then in the lexicographic order. We define a new partition of the state space by

$$\overline{\mathcal{L}}_m = \{(n_1, n_2; i_0, i_1, i_2) | n_2 = m\}, \quad m = 0, 1, 2, \ldots. \tag{6.1}$$

We denote by $\overline{\boldsymbol{Q}}$ the transition rate matrix of the chain corresponding to the arrangement above. Then, if we partition according to $\overline{\mathcal{L}}_m$'s, $\overline{\boldsymbol{Q}}$ is of a block-tridiagonal form

$$\overline{\boldsymbol{Q}} = \begin{pmatrix} \overline{\boldsymbol{B}}_0 & \overline{\boldsymbol{A}}_0 & & & \\ \overline{\boldsymbol{C}}_1 & \overline{\boldsymbol{B}} & \overline{\boldsymbol{A}} & & \\ & \overline{\boldsymbol{C}} & \overline{\boldsymbol{B}} & \overline{\boldsymbol{A}} & \\ & & \overline{\boldsymbol{C}} & \overline{\boldsymbol{B}} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{6.2}$$

where

$$\overline{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{O} & & & & \\ \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \otimes \boldsymbol{I}_2 & \boldsymbol{O} & & & \\ & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2 & \boldsymbol{O} & & \\ & & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2 & \boldsymbol{O} & \\ & & & \ddots & \ddots \end{pmatrix}, \tag{6.3}$$

$$\overline{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{T} \oplus \boldsymbol{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2 & & & \\ & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 & & \\ & & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2 & \\ & & & \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 & \ddots \\ & & & & \ddots \end{pmatrix}, \tag{6.4}$$

and

$$\overline{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & & & \\ & \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & & \\ & & \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & \\ & & & \boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \\ & & & & \ddots \end{pmatrix}. \tag{6.5}$$

We denote by $\overline{\boldsymbol{\pi}} = (\overline{\boldsymbol{\pi}}_0, \ \overline{\boldsymbol{\pi}}_1, \ \cdots)$ the stationary vector of $\{X(t)\}$ partitioned according to $\overline{\mathcal{L}}_m$'s.

We shall show that the conditions of Proposition 1 hold if we take $\eta = \overline{\eta}_2$ and if we define vectors $\overline{\boldsymbol{p}}$ and $\overline{\boldsymbol{q}}$ suitably. Similar to the preceding section, we first introduce several matrices and vectors including $\overline{\boldsymbol{q}}$, and then we check in Lemmas 6.1 $\sim$ 6.3 that the condition (4.2) $\sim$ (4.5) of the proposition hold. The geometric decay of the limiting vector $\overline{\boldsymbol{p}}$ will be proved in Lemma 6.5.

Let

$$\overline{\boldsymbol{K}} \equiv \left(\overline{\eta}_2^{-1}\overline{\boldsymbol{A}} + \overline{\boldsymbol{B}} + \overline{\eta}_2\overline{\boldsymbol{C}}\right) = \begin{pmatrix} \widehat{\boldsymbol{B}}_0 & \widehat{\boldsymbol{A}}_0 & & & \\ \widehat{\boldsymbol{C}}_1 & \widehat{\boldsymbol{B}} & \widehat{\boldsymbol{A}} & & \\ & \widehat{\boldsymbol{C}} & \widehat{\boldsymbol{B}} & \widehat{\boldsymbol{A}} & \\ & & \widehat{\boldsymbol{C}} & \widehat{\boldsymbol{B}} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{6.6}$$

Then from (6.3), (6.4) and (6.5)

$$\begin{aligned} \widehat{\boldsymbol{A}}_0 &= \boldsymbol{\gamma}_0\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2, & \widehat{\boldsymbol{A}} &= \boldsymbol{\gamma}_0\boldsymbol{\alpha} \otimes \boldsymbol{I}_1 \otimes \boldsymbol{I}_2, \\ \widehat{\boldsymbol{B}}_0 &= \boldsymbol{T} \oplus \boldsymbol{S}_2 + \overline{\eta}_2\boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2, & \widehat{\boldsymbol{B}} &= \boldsymbol{T} \oplus \boldsymbol{S}_1 \oplus \boldsymbol{S}_2 + \overline{\eta}_2\boldsymbol{I}_0 \otimes \boldsymbol{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2, \\ \widehat{\boldsymbol{C}}_1 &= \overline{\eta}_2^{-1}\boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \otimes \boldsymbol{I}_2 \quad \text{and} & \widehat{\boldsymbol{C}} &= \overline{\eta}_2^{-1}\boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \boldsymbol{I}_2. \end{aligned} \tag{6.7}$$

We define row vectors and column vectors as

$$\begin{aligned} &\overline{\boldsymbol{u}}_0 = \boldsymbol{\alpha}(\overline{\sigma}_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}, \quad \overline{\boldsymbol{u}}_1 = \boldsymbol{\beta}_1(-\boldsymbol{S}_1)^{-1}, \quad \overline{\boldsymbol{u}}_2 = \boldsymbol{\beta}_2(\overline{\sigma}_2\boldsymbol{I}_2 - \boldsymbol{S}_2)^{-1}, \\ &\overline{\boldsymbol{v}}_0 = (\overline{\sigma}_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}\boldsymbol{\gamma}_0 \quad \text{and} \quad \overline{\boldsymbol{v}}_2 = (\overline{\sigma}_2\boldsymbol{I}_2 - \boldsymbol{S}_2)^{-1}\boldsymbol{\gamma}_2 \,. \end{aligned} \tag{6.8}$$

Since $\overline{\omega}_2 = \overline{\sigma}_2$, the vector $\overline{\boldsymbol{u}}_2$ coincides with $\overline{\boldsymbol{w}}_2$ defined in (3.9). These vectors are solutions of equations of a type given in Lemma 4.1. They are related with LST of interarrival and service time distributions as follows:

$$\begin{aligned} &\overline{\boldsymbol{u}}_0\boldsymbol{\gamma}_0 = \boldsymbol{\alpha}\overline{\boldsymbol{v}}_0 = T^*(\overline{\sigma}_0) = \overline{\eta}_2, \quad \overline{\boldsymbol{u}}_1\boldsymbol{\gamma}_1 = \boldsymbol{\beta}_1\overline{\boldsymbol{v}}_1 = S_1^*(\overline{\sigma}_1) = \overline{\eta}_2^{-1}, \\ &\overline{\boldsymbol{u}}_0\overline{\boldsymbol{v}}_0 = -T^{*\prime}(\sigma_0), \quad \overline{\boldsymbol{u}}_1\overline{\boldsymbol{v}}_1 = -S_1^{*\prime}(\sigma_1) \quad \text{and} \quad \overline{\boldsymbol{u}}_2\boldsymbol{e} = -S_2^{*\prime}(0)\,. \end{aligned} \tag{6.9}$$

We define

$$\overline{\boldsymbol{q}} = \begin{pmatrix} \overline{\boldsymbol{v}}_0 \otimes \overline{\boldsymbol{v}}_2 \\ \overline{\eta}_2^{-1}\overline{\boldsymbol{v}}_0 \otimes \boldsymbol{e}_1 \otimes \overline{\boldsymbol{v}}_2 \\ \overline{\eta}_2^{-2}\overline{\boldsymbol{v}}_0 \otimes \boldsymbol{e}_1 \otimes \overline{\boldsymbol{v}}_2 \\ \vdots \end{pmatrix}.$$

Lemma 6.1. The vector $\overline{\boldsymbol{q}}$ is positive and satisfies $\overline{\boldsymbol{K}}\overline{\boldsymbol{q}} = \boldsymbol{0}$.

The proof is straightforward from Lemma 4.2.

Lemma 6.2. If $\overline{\eta}_1 < \overline{\eta}_2$, then there exists a positive vector $\overline{\boldsymbol{p}}$ such that

$$\overline{\boldsymbol{p}}\overline{\boldsymbol{K}} = \boldsymbol{0}, \quad \overline{\boldsymbol{p}}\boldsymbol{e} < \infty \quad \text{and} \quad \overline{\boldsymbol{p}}\,\overline{\boldsymbol{q}} < \infty.$$

Proof. As in Lemma 5.2 we consider the transformation $\overline{\boldsymbol{K}}_{\overline{D}} = \overline{\boldsymbol{D}}^{-1}\overline{\boldsymbol{K}}\,\overline{\boldsymbol{D}}$ of $\overline{\boldsymbol{K}}$ by $\overline{\boldsymbol{D}} = diag(\overline{\boldsymbol{q}})$:

$$\overline{\boldsymbol{K}}_{\overline{D}} = \begin{pmatrix} \widehat{\boldsymbol{D}}_0^{-1}\widehat{\boldsymbol{B}}_0\widehat{\boldsymbol{D}}_0 & \overline{\eta}_2^{-1}\widehat{\boldsymbol{D}}_0^{-1}\widehat{\boldsymbol{A}}_0\widehat{\boldsymbol{D}} & & & \\ \overline{\eta}_2\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{C}}_1\widehat{\boldsymbol{D}}_0 & \widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{B}}\widehat{\boldsymbol{D}} & \overline{\eta}_2^{-1}\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{A}}\widehat{\boldsymbol{D}} & & \\ & \overline{\eta}_2\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{C}}\widehat{\boldsymbol{D}} & \widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{B}}\widehat{\boldsymbol{D}} & \overline{\eta}_2^{-1}\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{A}}\widehat{\boldsymbol{D}} & \\ & & \overline{\eta}_2\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{C}}\widehat{\boldsymbol{D}} & \widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{B}}\widehat{\boldsymbol{D}} & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $\widehat{\boldsymbol{D}}_0 = diag(\overline{\boldsymbol{v}}_0 \otimes \overline{\boldsymbol{v}}_2)$ and $\widehat{\boldsymbol{D}} = diag(\overline{\boldsymbol{v}}_0 \otimes \boldsymbol{e}_1 \otimes \overline{\boldsymbol{v}}_2)$. $\overline{\boldsymbol{K}}_{\overline{D}}$ is a transition rate matrix of a QBD process with finitely many phases in each level. The ergodicity of $\overline{\boldsymbol{K}}_{\overline{D}}$ is proved as follows.

From Lemma 4.2, the vector $\widehat{\boldsymbol{\pi}} = (\overline{\boldsymbol{u}}_0 \otimes \overline{\boldsymbol{u}}_1 \otimes \overline{\boldsymbol{u}}_2)\ \widehat{\boldsymbol{D}}$ satisfies the equation

$$\widehat{\boldsymbol{\pi}} \left(\overline{\eta}_2^{-1} \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{A}} \widehat{\boldsymbol{D}} + \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{B}} \widehat{\boldsymbol{D}} + \overline{\eta}_2 \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{C}} \widehat{\boldsymbol{D}} \right) = \boldsymbol{0}. \tag{6.10}$$

A direct calculation shows that

$$\begin{aligned} &\widehat{\boldsymbol{\pi}} \left(\overline{\eta}_2^{-1} \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{A}} \widehat{\boldsymbol{D}} \right) \boldsymbol{e} - \widehat{\boldsymbol{\pi}} \left(\overline{\eta}_2 \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{C}} \widehat{\boldsymbol{D}} \right) \boldsymbol{e} \\ &= \overline{\eta}_2^{-1}\, \overline{\boldsymbol{u}}_0 \boldsymbol{\gamma}_0 \cdot \boldsymbol{\alpha} \overline{\boldsymbol{v}}_0 \cdot \overline{\boldsymbol{u}}_1 \boldsymbol{e}_1 \cdot \overline{\boldsymbol{u}}_2 \overline{\boldsymbol{v}}_2 - \overline{\eta}_2\, \overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0 \cdot \overline{\eta}_2\, \overline{\boldsymbol{u}}_1 \boldsymbol{\gamma}_1 \cdot \boldsymbol{\beta}_1 \boldsymbol{e}_1 \cdot \overline{\boldsymbol{u}}_2 \overline{\boldsymbol{v}}_2 \\ &= S_2^{*'}(\overline{\sigma}_2) \left\{ T^*(\overline{\sigma}_0)\, S_1^{*'}(0) - T^{*'}(\overline{\sigma}_0)\, S_1^*(0) \right\}. \end{aligned}$$

If $\overline{\eta}_1 < \overline{\eta}_2$, the quantity in the braces of the right most hand side of the above equation is positive from Lemma 8.7 in Appendix, and hence from Theorem 3.1.1 in [4] the Markov chain $\overline{\boldsymbol{K}}_{\overline{D}}$ is ergodic under the assumption of Theorem 3.2.

Thus there exists a positive vector $\overline{\boldsymbol{p}}_{\overline{D}}$ such that $\overline{\boldsymbol{p}}_{\overline{D}} \overline{\boldsymbol{K}}_{\overline{D}} = \boldsymbol{0}$ and $\overline{\boldsymbol{p}}_{\overline{D}} \boldsymbol{e} = 1$. Then,

$$\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}_{\overline{D}} \overline{\boldsymbol{D}}^{-1} \tag{6.11}$$

is the desired positive vector. In fact

$$\overline{\boldsymbol{p}}\, \overline{\boldsymbol{K}} = \overline{\boldsymbol{p}}_{\overline{D}} \overline{\boldsymbol{K}}_{\overline{D}} \overline{\boldsymbol{D}}^{-1} = \boldsymbol{0}, \qquad \overline{\boldsymbol{p}}\, \overline{\boldsymbol{q}} = \overline{\boldsymbol{p}}_{\overline{D}} \boldsymbol{e} = 1 < \infty \qquad \text{and} \qquad \overline{\boldsymbol{p}} \boldsymbol{e} < \overline{d}_1 \overline{\boldsymbol{p}}\, \overline{\boldsymbol{q}} < \overline{d}_1 < \infty,$$

where $\overline{d}_1$ is a positive number such that $\boldsymbol{e} < \overline{d}_1 \overline{\boldsymbol{q}}$. ∎

Lemma 6.3. For $\overline{\boldsymbol{p}}$ given in (6.11) we have $\overline{\eta}_2^{-1} \overline{\boldsymbol{p}} \overline{\boldsymbol{A}}\, \overline{\boldsymbol{q}} > \overline{\eta}_2\, \overline{\boldsymbol{p}}\, \overline{\boldsymbol{C}}\, \overline{\boldsymbol{q}}$.

Proof. We note that, from (6.6) and (6.7), $\overline{\boldsymbol{K}}$ is written in the form of a Kronecker sum

$$\overline{\boldsymbol{K}} = \overline{\boldsymbol{L}} \oplus (\boldsymbol{S}_2 + \overline{\eta}_2 \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 - \overline{\sigma}_2 \boldsymbol{I}_2). \tag{6.12}$$

with some matrix $\overline{\boldsymbol{L}}$. In order to derive a detailed structure of $\overline{\boldsymbol{p}}$, we shall decompose related matrices and vectors in a similar manner to (6.12). $\overline{\boldsymbol{K}}_{\overline{D}}$ is decomposed as

$$\overline{\boldsymbol{K}}_{\overline{D}} = \overline{\boldsymbol{L}}_{\overline{D}} \oplus (\widehat{\boldsymbol{S}}_2 + \overline{\eta}_2 \widehat{\boldsymbol{\gamma}}_2 \widehat{\boldsymbol{\beta}}_2 - \overline{\sigma}_2 \boldsymbol{I}_2), \tag{6.13}$$

where

$$\overline{\boldsymbol{L}}_{\overline{D}} = \begin{pmatrix} \widehat{\boldsymbol{T}} & \overline{\eta}_2^{-1} \widehat{\boldsymbol{\gamma}}_0 \widehat{\boldsymbol{\alpha}} \otimes \overline{\eta}_1 & & & \\ \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 & \widehat{\boldsymbol{T}} \oplus \boldsymbol{S}_1 & \overline{\eta}_2^{-1} \widehat{\boldsymbol{\gamma}}_0 \widehat{\boldsymbol{\alpha}} \otimes \boldsymbol{I}_1 & & \\ & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 & \widehat{\boldsymbol{T}} \oplus \boldsymbol{S}_1 & \overline{\eta}_2^{-1} \widehat{\boldsymbol{\gamma}}_0 \widehat{\boldsymbol{\alpha}} \otimes \boldsymbol{I}_1 & \\ & & \boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 & \widehat{\boldsymbol{T}} \oplus \boldsymbol{S}_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} - \overline{\sigma}_0 \boldsymbol{I}, \tag{6.14}$$

and

$$\widehat{\boldsymbol{T}} = diag(\overline{\boldsymbol{v}}_0)^{-1}\, \boldsymbol{T}\ diag(\overline{\boldsymbol{v}}_0), \qquad \widehat{\boldsymbol{S}}_2 = diag(\overline{\boldsymbol{v}}_2)^{-1}\, \boldsymbol{S}_2\ diag(\overline{\boldsymbol{v}}_2),$$

$$\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\ diag(\overline{\boldsymbol{v}}_0), \qquad \widehat{\boldsymbol{\beta}}_2 = \boldsymbol{\beta}_2\ diag(\overline{\boldsymbol{v}}_2),$$

$$\widehat{\boldsymbol{\gamma}}_0 = diag(\overline{\boldsymbol{v}}_0)^{-1}\, \boldsymbol{\gamma}_0, \qquad \text{and} \qquad \widehat{\boldsymbol{\gamma}}_2 = diag(\overline{\boldsymbol{v}}_2)^{-1}\, \boldsymbol{\gamma}_2.$$

The matrix $\overline{\boldsymbol{L}}$ in (6.12) is given as $\overline{\boldsymbol{L}}_{\overline{D}}$ in (6.14) by removing hats. $\overline{\boldsymbol{q}}$ and $\overline{\boldsymbol{D}}$ are decomposed as $\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}_{\overline{L}} \otimes \overline{\boldsymbol{v}}_2$ and $\overline{\boldsymbol{D}} = \overline{\boldsymbol{D}}_{\overline{L}} \otimes diag(\overline{\boldsymbol{v}}_2)$, where

$$\overline{\boldsymbol{q}}_{\overline{L}} = \begin{pmatrix} \overline{\boldsymbol{v}}_0 \\ \overline{\eta}_2^{-1}\overline{\boldsymbol{v}}_0 \otimes \boldsymbol{e}_1 \\ \overline{\eta}_2^{-2}\overline{\boldsymbol{v}}_0 \otimes \boldsymbol{e}_1 \\ \vdots \end{pmatrix} \qquad \text{and} \qquad \overline{\boldsymbol{D}}_{\overline{L}} = diag(\overline{\boldsymbol{q}}_{\overline{L}}).$$

It is easily checked that $\overline{\boldsymbol{q}}_{\overline{L}} > 0$ and $\overline{\boldsymbol{L}}\,\overline{\boldsymbol{q}}_{\overline{L}} = \boldsymbol{0}$, or equivalently $\overline{\boldsymbol{L}}_{\overline{D}}\boldsymbol{e} = \boldsymbol{0}$. Hence $\overline{\boldsymbol{L}}_{\overline{D}}$ is a transition rate matrix of a QBD process with a finite number of phases in each level. The ergodicity of $\overline{\boldsymbol{L}}_{\overline{D}}$ is clear from that of $\overline{\boldsymbol{K}}_{\overline{D}}$, and there exists a stationary probability vector $\overline{\boldsymbol{p}}_{\overline{LD}}$ of the chain:

$$\overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{L}}_{\overline{D}} = \boldsymbol{0} \qquad \text{and} \qquad \overline{\boldsymbol{p}}_{\overline{LD}}\,\boldsymbol{e} = 1.$$

Using this $\overline{\boldsymbol{p}}_{\overline{LD}}$, we can decompose $\overline{\boldsymbol{p}}$ and $\overline{\boldsymbol{p}}_{\overline{D}}$ as follows:

$$\overline{\boldsymbol{p}}_{\overline{D}} = (\overline{\boldsymbol{u}}_2\overline{\boldsymbol{v}}_2)^{-1}\,\overline{\boldsymbol{p}}_{\overline{LD}} \otimes \overline{\boldsymbol{u}}_2\; diag(\overline{\boldsymbol{v}}_2) \qquad \text{and} \qquad \overline{\boldsymbol{p}} = (\overline{\boldsymbol{u}}_2\overline{\boldsymbol{v}}_2)^{-1}\,\overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{D}}_{\overline{L}}^{-1} \otimes \overline{\boldsymbol{u}}_2.$$

Now we evaluate the both sides of the inequality of the lemma. Since $\overline{\boldsymbol{C}}\,\overline{\boldsymbol{q}} = \overline{\eta}_2^{-1}\,\overline{\boldsymbol{q}}_{\overline{L}} \otimes \boldsymbol{\gamma}_2$, we have

$$\overline{\eta}_2\,\overline{\boldsymbol{p}}\,\overline{\boldsymbol{C}}\,\overline{\boldsymbol{q}} \;=\; (\overline{\boldsymbol{u}}_2\overline{\boldsymbol{v}}_2)^{-1}\,\overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{D}}_{\overline{L}}^{-1}\,\overline{\boldsymbol{q}}_{\overline{L}} \cdot \overline{\boldsymbol{u}}_2\boldsymbol{\gamma}_2 \;=\; \overline{\eta}_2^{-1}\,(\overline{\boldsymbol{u}}_2\overline{\boldsymbol{v}}_2)^{-1}, \tag{6.15}$$

where we use the relation $\overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{D}}_{\overline{L}}^{-1}\,\overline{\boldsymbol{q}}_{\overline{L}} = \overline{\boldsymbol{p}}_{\overline{LD}}\boldsymbol{e} = 1$. On the other hand, since

$$\overline{\eta}_2^{-1}\overline{\boldsymbol{D}}^{-1}\overline{\boldsymbol{A}}\overline{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1 \otimes \boldsymbol{e}_2 \\ \boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1 \otimes \boldsymbol{e}_2 \\ \vdots \end{pmatrix},$$

we have

$$\overline{\eta}_2^{-1}\,\overline{\boldsymbol{p}}\,\overline{\boldsymbol{A}}\,\overline{\boldsymbol{q}} \;=\; \overline{\eta}_2^{-1}\,\overline{\boldsymbol{p}}_{\overline{D}}\,\overline{\boldsymbol{D}}^{-1}\,\overline{\boldsymbol{A}}\,\overline{\boldsymbol{q}} \;=\; \overline{\boldsymbol{p}}_{\overline{LD}} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1 \\ \boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1 \\ \vdots \end{pmatrix}. \tag{6.16}$$

Note that $\boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1 = (\boldsymbol{I}_0 \otimes \boldsymbol{\gamma}_1\boldsymbol{\beta}_1)(\boldsymbol{e}_0 \otimes \boldsymbol{e}_1)$. This means that the ith element of $\boldsymbol{e}_0 \otimes \boldsymbol{\gamma}_1$ is the rate that the Markov chain $\overline{\boldsymbol{L}}_{\overline{D}}$ at state (n, i) goes down to level $(n-1)$. Hence the quantity (6.16) is the rate that the chain $\overline{\boldsymbol{L}}_{\overline{D}}$ goes one level down in the steady state. From the balance equation, this rate is equal to the rate that the chain goes one level up:

$$\overline{\eta}_2^{-1}\overline{\boldsymbol{p}}\overline{\boldsymbol{A}}\overline{\boldsymbol{q}} = \overline{\boldsymbol{p}}_{\overline{LD}} \begin{pmatrix} \widehat{\gamma}_0 \\ \widehat{\gamma}_0 \otimes \boldsymbol{e}_1 \\ \widehat{\gamma}_0 \otimes \boldsymbol{e}_1 \\ \vdots \end{pmatrix} = \overline{\boldsymbol{p}}_{\overline{LD}}\boldsymbol{E}_{\overline{L}}\widehat{\gamma}_0, \quad \text{with} \quad \boldsymbol{E}_{\overline{L}} = \begin{pmatrix} \boldsymbol{I}_0 \\ \boldsymbol{I}_0 \otimes \boldsymbol{e}_1 \\ \boldsymbol{I}_0 \otimes \boldsymbol{e}_1 \\ \vdots \end{pmatrix}. \tag{6.17}$$

By postmultiplying $\boldsymbol{E}_{\overline{L}}$ to the equation $\boldsymbol{0} = \overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{L}}_{\overline{D}}$, we have

$$\begin{aligned} \boldsymbol{0} \;=\; \overline{\boldsymbol{p}}_{\overline{LD}}\,\overline{\boldsymbol{L}}_{\overline{D}}\,\boldsymbol{E}_{\overline{L}} \;&=\; \overline{\boldsymbol{p}}_{\overline{LD}} \begin{pmatrix} \widehat{\boldsymbol{T}} + \overline{\sigma}_2\boldsymbol{I}_2 + \overline{\eta}_2^{-1}\widehat{\gamma}_0\boldsymbol{\alpha} \\ (\widehat{\boldsymbol{T}} + \overline{\sigma}_2\boldsymbol{I}_2 + \overline{\eta}_2^{-1}\widehat{\gamma}_0\boldsymbol{\alpha}) \otimes \boldsymbol{e}_1 \\ (\widehat{\boldsymbol{T}} + \overline{\sigma}_2\boldsymbol{I}_2 + \overline{\eta}_2^{-1}\widehat{\gamma}_0\boldsymbol{\alpha}) \otimes \boldsymbol{e}_1 \\ \vdots \end{pmatrix} \\ &=\; \overline{\boldsymbol{p}}_{\overline{LD}}\,\boldsymbol{E}_{\overline{L}}\,(\widehat{\boldsymbol{T}} + \overline{\sigma}_2\boldsymbol{I}_2 + \overline{\eta}_2^{-1}\widehat{\gamma}_0\boldsymbol{\alpha}). \end{aligned}$$

This indicates that $\overline{\boldsymbol{p}}_{\overline{LD}}\, \boldsymbol{E}_{\overline{L}}$ can be regarded as the stationary probability vector of a Markov chain with transition rate matrix $\widehat{\boldsymbol{T}} + \overline{\sigma}_2 \boldsymbol{I}_2 + \overline{\eta}_2^{-1}\widehat{\boldsymbol{\gamma}}_0 \boldsymbol{\alpha}$, and is given by

$$\overline{\boldsymbol{p}}_{\overline{LD}}\, \boldsymbol{E}_{\overline{L}} \;=\; (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1}\, \overline{\boldsymbol{u}}_0\ diag(\overline{\boldsymbol{v}}_0).$$

Thus, from (6.17), we have

$$\overline{\eta}_2^{-1}\, \overline{\boldsymbol{p}}\, \overline{\boldsymbol{A}}\, \overline{\boldsymbol{q}} \;=\; (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1}\, \overline{\boldsymbol{u}}_0\ diag(\overline{\boldsymbol{v}}_0)\, \widehat{\boldsymbol{\gamma}}_0 \;=\; (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1} \cdot \overline{\eta}_2. \tag{6.18}$$

Therefore, from (6.15) and (6.18) we have

$$\begin{aligned}\overline{\eta}_2^{-1}\, \overline{\boldsymbol{p}}\, \overline{\boldsymbol{A}}\, \overline{\boldsymbol{q}} - \overline{\eta}_2\, \overline{\boldsymbol{p}}\, \overline{\boldsymbol{C}}\, \overline{\boldsymbol{q}} \;&=\; (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1} (\overline{\boldsymbol{u}}_2 \overline{\boldsymbol{v}}_2)^{-1} \left\{ \overline{\eta}_2\, (\overline{\boldsymbol{u}}_2 \overline{\boldsymbol{v}}_2)^{-1} - (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1}\, \overline{\eta}_2^{-1} \right\} \\ &=\; (\overline{\boldsymbol{u}}_0 \overline{\boldsymbol{v}}_0)^{-1} (\overline{\boldsymbol{u}}_2 \overline{\boldsymbol{v}}_2)^{-1} \left\{ -T^*(\overline{\sigma}_0) S_2^{*'}(\overline{\sigma}_2) + T^{*'}(\overline{\sigma}_0) S_2^*(\overline{\sigma}_2) \right\}.\end{aligned}$$

From Lemma 8.8 the right hand side is positive, and this proves the lemma. ∎

Lemma 6.4. If $\eta_1 < \overline{\eta}_2$, then $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{q}} < \infty$.

Proof. Since $\overline{\boldsymbol{\pi}}_1 \leq \sum_{m=0}^{\infty} \overline{\boldsymbol{\pi}}_m$, we have

$$\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{q}} \leq \sum_{m=0}^{\infty} \overline{\boldsymbol{\pi}}_m \overline{\boldsymbol{q}}. \tag{6.19}$$

The behavior of the first stage of our tandem queueing system is not affected by the second one, and the stationary marginal probability vector $\sum_{m=0}^{\infty} \overline{\boldsymbol{\pi}}_m$ of the first stage decays geometrically with rate η_1 as $n_1 \to \infty$ [6]. Since $\overline{\boldsymbol{q}}$ decays with rate $\overline{\eta}_2^{-1}$, the inner product $\sum_{m=0}^{\infty} \overline{\boldsymbol{\pi}}_m \overline{\boldsymbol{q}}$ is finite if $\eta_1 / \overline{\eta}_2 < 1$. ∎

To see the asymptotic form of $\overline{\boldsymbol{p}}$, we consider a subpartition of each $\overline{\mathcal{L}}_m$,

$$\bar{l}_n = \{(n_1; i_0, i_1, i_2) | n_1 = n\}, \quad n = 0, 1, 2, \ldots,$$

and denote by $\overline{\boldsymbol{p}} = (\overline{\boldsymbol{p}}(0),\ \overline{\boldsymbol{p}}(1),\ \cdots)$ the row vector $\overline{\boldsymbol{p}}$ partitioned according to $\bar{l}_n$'s.

Lemma 6.5. If $\overline{\eta}_1 < 1$, then

$$\overline{\boldsymbol{p}}(n_1) \sim \overline{G}'\, \overline{\eta}_1^{n_1}\, \overline{\boldsymbol{w}}_0 \otimes \overline{\boldsymbol{w}}_1 \otimes \overline{\boldsymbol{w}}_2, \qquad \text{as} \quad n_1 \to \infty,$$

where $\overline{G}'$ is a constant and $\overline{\boldsymbol{w}}_0$, $\overline{\boldsymbol{w}}_1$ $(= \overline{\boldsymbol{w}}_{11})$ and $\overline{\boldsymbol{w}}_2$ $(= \overline{\boldsymbol{w}}_{21})$ are vectors defined in (3.9).

Proof. In a similar manner to the proof of Lemma 5.4, we apply the matrix analytic theory by Neuts [4] to the rate matrix $\overline{\boldsymbol{K}}_{\overline{D}}$. Let $\widehat{\boldsymbol{R}}_{\overline{D}}$ be the rate matrix of $\overline{\boldsymbol{K}}_{\overline{D}}$ satisfying

$$\widehat{\boldsymbol{D}}^{-1}\, \widehat{\boldsymbol{A}}\, \overline{\boldsymbol{D}} + \widehat{\boldsymbol{R}}_{\overline{D}}\, \widehat{\boldsymbol{D}}^{-1}\, \widehat{\boldsymbol{B}}\, \widehat{\boldsymbol{D}} + \widehat{\boldsymbol{R}}_{\overline{D}}^2\, \widehat{\boldsymbol{D}}^{-1}\, \widehat{\boldsymbol{C}}\, \widehat{\boldsymbol{D}} = \boldsymbol{O}. \tag{6.20}$$

Since $\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}_{\overline{D}}\, \overline{\boldsymbol{D}}$, we know that

$$\overline{\boldsymbol{p}}(n_1) = \overline{\boldsymbol{p}}(1)\, \widehat{\boldsymbol{D}}\, \widehat{\boldsymbol{R}}_{\overline{D}}^{n_1}\, \widehat{\boldsymbol{D}}^{-1} \sim \overline{G}''\, \widehat{\eta}^{n_1}\, \widehat{\boldsymbol{x}}_{\overline{D}}\, \widehat{\boldsymbol{D}}^{-1}, \qquad \text{as} \quad n_1 \to \infty,$$

where G'' is a constant, $\widehat{\eta}$ is the Perron-Frobenius eigenvalue of $\widehat{\boldsymbol{R}}_{\overline{D}}$ and $\widehat{\boldsymbol{x}}_{\overline{D}}$ is the corresponding left eigenvector: $\widehat{\boldsymbol{x}}_{\overline{D}}\widehat{\boldsymbol{R}}_{\overline{D}} = \widehat{\eta}\,\widehat{\boldsymbol{x}}_{\overline{D}}$. Then by premultiplying $\widehat{\boldsymbol{x}}_{\overline{D}}$ to the left hand side of (6.20), we have

$$\begin{aligned} \mathbf{0} &= \widehat{\boldsymbol{x}}_{\overline{D}}\widehat{\boldsymbol{D}}^{-1}\left(\widehat{\eta}^{-1},\widehat{\boldsymbol{A}}+\widehat{\boldsymbol{B}}+\widehat{\eta}\widehat{\boldsymbol{C}}\right) \\ &= \widehat{\boldsymbol{x}}_{\overline{D}}\widehat{\boldsymbol{D}}^{-1}\left[\left(\widehat{\eta}^{-1}\gamma_0\boldsymbol{\alpha}+\boldsymbol{T}\right)\oplus\left(\overline{\eta}_2^{-1}\widehat{\eta}\gamma_1\boldsymbol{\beta}_1+\boldsymbol{S}_1\right)\oplus\left(\overline{\eta}_2\gamma_2\boldsymbol{\beta}_2+\boldsymbol{S}_2\right)\right]. \end{aligned}$$

From an extended version of Lemma 4.2, in order that the positive vector $\widehat{\boldsymbol{x}}_{\overline{D}}\widehat{\boldsymbol{D}}^{-1}$ satisfies the above equation, there exit s_0, s_1 and s_2 such that

$$\begin{cases} T^*(s_0) = \widehat{\eta}, \\ S^*_{1j}(s_1) = \overline{\eta}_2\widehat{\eta}^{-1}, \\ S^*_{2j}(s_2) = \overline{\eta}_2^{-1}, \\ s_0+s_1+s_2 = 0. \end{cases}$$

These equations are the same as (3.8) in the single server case, and hence $s_0 = \overline{\omega}_0$, $s_1 = \overline{\omega}_1$, $s_2 = \overline{\omega}_2$ and $\widehat{\eta} = \overline{\eta}_1$ since $\widehat{\eta}$ has to be less than 1. Again from Lemma 4.2, we know that

$$\widehat{\boldsymbol{x}}_{\overline{D}}\widehat{\boldsymbol{D}}^{-1} = \overline{G}'''\,\overline{\boldsymbol{w}}_0\otimes\overline{\boldsymbol{w}}_1\otimes\overline{\boldsymbol{w}}_2,$$

where $\overline{G}'''$ is a constant. This proves the lemma. ∎

Lemma 6.5 proves (3.10) of Theorem 3.2, and this completes the proof of the theorem.

7. Extension to Multi-server Case

The proofs in Sections 5 and 6 for Theorems 3.1 and 3.2 can be extended straightforwardly to the case with heterogeneous multiple servers in each stage. In this section, we give brief comments on the proofs of the theorems.

The state space of the Markov chain $\{X(t)\}$ is divided in a similar manner to (5.1), but here i_1 should be regarded as a vector $(i_{11},\ldots,i_{1c_1})$ and i_2 as a vector $(i_{11},\ldots,i_{1c_1})$. In this case matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in (4.1) are of the form

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{00} & & & & & & \\ & \boldsymbol{A}_{11} & & & & & \\ & & \ddots & & & & \\ & & & \boldsymbol{A}_{c_1c_1} & & & \\ & & & & \boldsymbol{A}_{c_1c_1} & & \\ & & & & & \boldsymbol{A}_{c_1c_1} & \\ & & & & & & \ddots \end{pmatrix}, \qquad (7.1)$$

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{00} & & & & & & \\ \boldsymbol{B}_{10} & \boldsymbol{B}_{11} & & & & & \\ & \ddots & \ddots & & & & \\ & & \boldsymbol{B}_{c_1c_1-1} & \boldsymbol{B}_{c_1c_1} & & & \\ & & & \boldsymbol{B}_{c_1+1c_1} & \boldsymbol{B}_{c_1c_1} & & \\ & & & & \boldsymbol{B}_{c_1+1c_1} & \boldsymbol{B}_{c_1c_1} & \\ & & & & & \ddots & \ddots \end{pmatrix}, \qquad (7.2)$$

and

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{C}_{01} & & & & & \\ & \boldsymbol{O} & \boldsymbol{C}_{12} & & & & \\ & & \ddots & \ddots & & & \\ & & & \boldsymbol{O} & \boldsymbol{C}_{c_1c_1+1} & & \\ & & & & \boldsymbol{O} & \boldsymbol{C}_{c_1c_1+1} & \\ & & & & & \boldsymbol{O} & \ddots \\ & & & & & & \ddots \end{pmatrix}, \tag{7.3}$$

where

$$\begin{aligned} \boldsymbol{A}_{00} &= \gamma_0\boldsymbol{\alpha} \otimes (\boldsymbol{I}_{11} \otimes \cdots \otimes \boldsymbol{I}_{1c_1}), \\ \boldsymbol{A}_{c_1c_1} &= \gamma_0\boldsymbol{\alpha} \otimes (\boldsymbol{I}_{11} \otimes \cdots \otimes \boldsymbol{I}_{1c_1}) \otimes (\boldsymbol{I}_{21} \otimes \cdots \otimes \boldsymbol{I}_{2c_2}), \\ \boldsymbol{B}_{00} &= \boldsymbol{T} \oplus (\boldsymbol{S}_{11} \oplus \cdots \oplus \boldsymbol{S}_{1c_1}), \\ \boldsymbol{B}_{c_1c_1} &= \boldsymbol{T} \oplus (\boldsymbol{S}_{11} \oplus \cdots \oplus \boldsymbol{S}_{1c_1}) \oplus (\boldsymbol{S}_{21} \oplus \cdots \oplus \boldsymbol{S}_{2c_2}), \\ \boldsymbol{B}_{c_1+1c_1} &= \boldsymbol{I}_0 \otimes (\boldsymbol{I}_{11} \otimes \cdots \otimes \boldsymbol{I}_{1c_1}) \otimes (\boldsymbol{\gamma}_{21}\boldsymbol{\beta}_{21} \oplus \cdots \oplus \boldsymbol{\gamma}_{2c_2}\boldsymbol{\beta}_{2c_2}), \\ \boldsymbol{C}_{c_1c_1+1} &= \boldsymbol{I}_0 \otimes (\boldsymbol{\gamma}_{11}\boldsymbol{\beta}_{11} \oplus \cdots \oplus \boldsymbol{\gamma}_{1c_1}\boldsymbol{\beta}_{1c_1}) \otimes (\boldsymbol{I}_{21} \otimes \cdots \otimes \boldsymbol{I}_{2c_2}), \end{aligned}$$

and other matrices are very complicated for writing down explicitly here since $\mathcal{L}_m$ for $0 < m < c_2$ consists of states corresponding to various combinations of busy and idle servers.

The proof of Theorem 3.1 for the multi-server case is parallel to that for the single server case given in Section 5. In this section, we use some symbols without explicit definitions if the meaning of them are easily understood from the corresponding ones defined for the single server case. For the solution $(\eta_1, \sigma_0, \sigma_{11}, \ldots, \sigma_{1c_1})$ of the equation (3.2), we define vectors

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{\alpha}(\sigma_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}, & & \\ \boldsymbol{u}_{1j} &= \boldsymbol{\beta}_{1j}(\sigma_{1j}\boldsymbol{I}_{1j} - \boldsymbol{S}_{1j})^{-1}, & & j = 1, 2, \ldots, c_1, \\ \boldsymbol{u}_{2j} &= \boldsymbol{\beta}_{2j}(-\boldsymbol{S}_{2j})^{-1}, & & j = 1, 2, \ldots, c_1, \\ \boldsymbol{v}_0 &= (\sigma_0\boldsymbol{I}_0 - \boldsymbol{T})^{-1}\boldsymbol{\gamma}_0\,, & & \\ \text{and} \quad \boldsymbol{v}_{1j} &= (\sigma_{1j}\boldsymbol{I}_{1j} - \boldsymbol{S}_{1j})^{-1}\boldsymbol{\gamma}_{1j}\,, & & j = 1, 2, \ldots, c_1. \end{aligned} \tag{7.4}$$

These vectors satisfy similar relations to (5.7). The vector $\boldsymbol{q}$ is defined as

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{\underline{1}} \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{\underline{2}} \\ \vdots \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{\underline{c_2}} \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{21} \otimes \boldsymbol{e}_{22} \otimes \cdots \otimes \boldsymbol{e}_{2c_2} \\ \boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{21}\boldsymbol{e}_{22} \otimes \otimes \cdots \otimes \boldsymbol{e}_{2c_2} \\ \vdots \end{pmatrix},$$

where $\boldsymbol{e}_{\underline{n}}$ is the column vector of the same order as the number of states in l_n with all entries equal to 1.

For the existence of $\boldsymbol{p}$, we extend Lemma 5.2 to multi-server case. That is, if we put

$$\tilde{\boldsymbol{D}} = diag(\boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \cdots \otimes \boldsymbol{v}_{1c_1} \otimes \boldsymbol{e}_{21} \otimes \cdots \otimes \boldsymbol{e}_{2c_2}),$$

then $\boldsymbol{p}$ exists iff

$$\tilde{\boldsymbol{\pi}}\{\tilde{\boldsymbol{D}}^{-1}(\eta_1^{-1}\boldsymbol{C}_{c_1c_1})\tilde{\boldsymbol{D}}\}\boldsymbol{e} < \tilde{\boldsymbol{\pi}}(\tilde{\boldsymbol{D}}^{-1}\boldsymbol{B}_{c_1c_1}\tilde{\boldsymbol{D}})\boldsymbol{e}.$$

By definitions, this inequality is equivalent to the following one:

$$\frac{1}{\eta_1}\sum_{j=1}^{c_1}\frac{1}{S_{1j}^{*'}(\sigma_{1j})} > \sum_{j=1}^{c_2}\frac{1}{S_{2j}^{*'}(0)}. \tag{7.5}$$

If $\eta_2 < 1$ the inequality (7.5) holds by Lemma 8.5 in Appendix, and hence Lemma 5.2 is extended to multi-server case.

A similar argument to the above shows that Lemma 5.3 holds for multi-server case. In fact, if we let

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \otimes \boldsymbol{e}_{\underline{1}} \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \otimes \boldsymbol{e}_{\underline{2}} \\ \vdots \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \otimes \boldsymbol{e}_{\underline{c_2}} \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \otimes \boldsymbol{e}_{21} \otimes \boldsymbol{e}_{22} \otimes \cdots \otimes \boldsymbol{e}_{2c_2} \\ \boldsymbol{I}_0 \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1} \otimes \boldsymbol{e}_{21} \otimes \boldsymbol{e}_{22} \otimes \cdots \otimes \boldsymbol{e}_{2c_2} \\ \vdots \end{pmatrix},$$

then

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{E}(\boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1}), \\ \boldsymbol{KE} &= \boldsymbol{E}(\eta_1^{-1}\gamma_0\boldsymbol{\alpha} + \boldsymbol{T}) \oplus (\eta_1\gamma_{11}\boldsymbol{\beta}_{11} + \boldsymbol{S}_{11}) \oplus (\eta_1\gamma_{12}\boldsymbol{\beta}_{12} + \boldsymbol{S}_{12}) \oplus \cdots \\ &\qquad \oplus (\eta_1\gamma_{1c_1}\boldsymbol{\beta}_{1c_1} + \boldsymbol{S}_{1c_1}), \\ \boldsymbol{AE} &= \boldsymbol{E}(\gamma_0\boldsymbol{\alpha} \otimes \boldsymbol{I}_{11} \otimes \boldsymbol{I}_{12} \otimes \cdots \otimes \boldsymbol{I}_{1c_1}), \\ \boldsymbol{CE} &= \boldsymbol{E}\{\boldsymbol{I}_0 \otimes (\gamma_{11}\boldsymbol{\beta}_{11} \oplus \gamma_{12}\boldsymbol{\beta}_{12} \oplus \cdots \oplus \gamma_{1c_1}\boldsymbol{\beta}_{1c_1})\}. \end{aligned}$$

For $\boldsymbol{p}$, we postmultiply $\boldsymbol{E}$ to the equality $\boldsymbol{pK} = \boldsymbol{0}$ to have

$$\boldsymbol{pE}\{(\eta_1^{-1}\gamma_0\boldsymbol{\alpha} + \boldsymbol{T}) \oplus (\eta_1\gamma_{11}\boldsymbol{\beta}_{11} + \boldsymbol{S}_{11}) \oplus (\eta_1\gamma_{12}\boldsymbol{\beta}_{12} + \boldsymbol{S}_{12}) \oplus \cdots \oplus (\eta_1\gamma_{1c_1}\boldsymbol{\beta}_{1c_1} + \boldsymbol{S}_{1c_1})\} = \boldsymbol{0}.$$

From Lemma 4.2, $\boldsymbol{pE}$ is a constant multiple of $\boldsymbol{u}_0 \otimes \boldsymbol{u}_{11} \otimes \cdots \otimes \boldsymbol{u}_{1c_1}$. Let H' be the multiplicative constant, then we have

$$\begin{aligned} \eta_1^{-1}\boldsymbol{pAq} &= \boldsymbol{pE}(\gamma_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1}) \\ &= H'\boldsymbol{u}_0\gamma_0\prod_{j=1}^{c_1}\boldsymbol{u}_{1j}\boldsymbol{v}_{1j} = H'\eta_1\prod_{j=1}^{c_1}\{-S_{1j}^{*'}(\sigma_{1j})\}, \\ \eta_1\boldsymbol{pCq} &= \eta_1\boldsymbol{pE}\{\boldsymbol{I}_0 \otimes (\gamma_{11}\boldsymbol{\beta}_{11} \oplus \gamma_{12}\boldsymbol{\beta}_{12} \oplus \cdots \oplus \gamma_{1c_1}\boldsymbol{\beta}_{1c_1})\}(\boldsymbol{v}_0 \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{12} \otimes \cdots \otimes \boldsymbol{v}_{1c_1})\} \\ &= H'\frac{1}{\eta_1}\boldsymbol{u}_0\boldsymbol{v}_0\sum_{j=1}^{c_1}\prod_{k\neq j}\boldsymbol{u}_{1k}\boldsymbol{v}_{1k} = H'\frac{1}{\eta_1}T^{*'}(\sigma_0)\prod_{j=1}^{c_1}\{-S_{1j}^{*'}(\sigma_{1j})\}\sum_{k=1}^{c_1}\left\{\frac{1}{S_{1k}^{*'}(\sigma_{1k})}\right\}. \end{aligned}$$

Hence the inequality of Lemma 5.3 is equivalent to

$$\eta_1 < \frac{1}{\eta_1} T^{*'}(\sigma_0) \sum_{j=1}^{c_1} \left\{ \frac{1}{S_{1j}^{*'}(\sigma_{1j})} \right\}.$$

This holds from Lemma 8.6, and hence Theorem 3.1 is proved for the multi-server case.

We can derive a proof of Theorem 3.2 for the multi-server case in a similar manner. We omit the proof here.

8. Appendix

In this section, we prove various properties of the solutions of the four key systems of equations (3.2), (3.3), (3.7) and (3.8) given in Section 3.

Lemma 8.1. The system of equations (3.2) has two solutions, one of which is $(1, 0, 0, \ldots, 0)$. For the other solution $(\eta_1, \sigma_0, \sigma_{11}, \ldots, \sigma_{1c_1})$, $\eta_1 < 1$, $\sigma_0 > 0$ and $\sigma_{1j} < 0$.

Proof. From the second equation of (3.2), we have

$$S_{1j}^*(s_{1j}) = S_{11}^*(s_{11}), \qquad j = 2, 3, \ldots, c_1. \tag{8.1}$$

Since $S_{1j}^*(s_{1j})$ is a monotone decreasing function, the relation (8.1) defines s_{1j} as a function of s_{11}. Then, from the third equation of (3.2), $s_0 = -s_{11} - s_{12} - \cdots - s_{1c_1}$ is also interpreted as a function of s_{11}. Since s_0 is a monotone decreasing function of s_{11}, we may take s_0 as an independent variable and regard $s_{11}, s_{12}, \ldots, s_{1c_1}$ as monotone functions of s_0. For brevity of notations, we introduce a function

$$U_1^*(-s_0) = S_{11}^*(s_{11}(s_0)). \tag{8.2}$$

Then the system of equations (3.2) can be rewritten as

$$\begin{cases} T^*(s_0)U_1^*(-s_0) = 1, \\ S_{1j}^*(s_{1j}) = U_1^*(-s_0), \qquad j = 1, 2, \ldots, c_1, \\ s_0 + s_{11} + \cdots + s_{1c_1} = 0. \end{cases} \tag{8.3}$$

Now we show that the function $U_1^*(-s_0)$ is a logarithmic-convex function, that is, $\log U_1^*(-s_0)$ is a convex function and satisfies

$$U_1^{*''}(-s_0)U_1^*(-s_0) - \{U_1^{*'}(-s_0)\}^2 > 0. \tag{8.4}$$

The first differentiation $U_1^{*'}(-s_0)$ can be obtained as follows. Differentiating both sides of the second equation of (8.3) by s_0, we have

$$-U_1^{*'}(-s_0) = \frac{d}{ds_0} S_{1j}^*(s_{1j}) = \frac{ds_{1j}}{ds_0} \frac{d}{ds_{1j}} S_{1j}^*(s_{1j}) = \frac{ds_{1j}}{ds_0} S_{1j}^{*'}(s_{1j}), \qquad j = 1, 2, \ldots, c_1. \tag{8.5}$$

It follows from the third equation of (8.3) that

$$1 + \frac{ds_{11}}{ds_0} + \frac{ds_{12}}{ds_0} + \cdots + \frac{ds_{1c_1}}{ds_0} = 0.$$

Substituting (8.5), we have

$$\frac{U_1^{*'}(-s_0)}{S_{11}^{*'}(s_{11})}+\frac{U_1^{*'}(-s_0)}{S_{12}^{*'}(s_{12})}+\cdots+\frac{U_1^{*'}(-s_0)}{S_{1c_1}^{*'}(s_{1c_1})}=1, \tag{8.6}$$

and hence

$$U_1^{*'}(-s_0)=\left(\sum_{j=1}^{c_1}\frac{1}{S_{1j}^{*'}(s_{1j})}\right)^{-1}. \tag{8.7}$$

For $U_1^{*''}(-s_0)$, differentiating both sides of (8.6) by s_0 we have

$$\begin{aligned} 0 &= -U_1^{*''}(-s_0)\left\{\frac{1}{S_{11}^{*'}(s_{11})}+\frac{1}{S_{12}^{*'}(s_{12})}+\cdots+\frac{1}{S_{1c_1}^{*'}(s_{1c_1})}\right\} \\ &\quad -U_1^{*'}(-s_0)\left\{\frac{S_{11}^{*''}(s_{11})}{S_{11}^{*'}(s_{11})^2}\frac{ds_{11}}{ds_0}+\frac{S_{12}^{*''}(s_{12})}{S_{12}^{*'}(s_{12})^2}\frac{ds_{12}}{ds_0}+\cdots+\frac{S_{1c_1}^{*''}(s_{1c_1})}{S_{1c_1}^{*'}(s_{1c_1})^2}\frac{ds_{1c_1}}{ds_0}\right\} \\ &= -\frac{U_1^{*''}(-s_0)}{U_1^{*'}(-s_0)}+\{U_1^{*'}(-s_0)\}^2\left\{\frac{S_{11}^{*''}(s_{11})}{S_{11}^{*'}(s_{11})^3}+\frac{S_{12}^{*''}(s_{12})}{S_{12}^{*'}(s_{12})^3}+\cdots+\frac{S_{1c_1}^{*''}(s_{1c_1})}{S_{1c_1}^{*'}(s_{1c_1})^3}\right\}. \end{aligned}$$

And hence

$$U_1^{*''}(-s_0)=\{U_1^{*'}(-s_0)\}^3\sum_{j=1}^{c_1}\frac{S_{1j}^{*''}(s_{1j})}{S_{1j}^{*'}(s_{1j})^3}.$$

Therefore,

$$\begin{aligned} U_1^{*''}(-s_0)U_1^*(-s_0)-\{U_1^{*'}(-s_0)\}^2 &= \{U_1^{*'}(-s_0)\}^3\sum_{j=1}^{c_1}\frac{S_{1j}^{*''}(s_{1j})U_1^*(-s_0)}{S_{1j}^{*'}(s_{1j})^3}-\{U_1^{*'}(-s_0)\}^2 \\ &= -\{U_1^{*'}(-s_0)\}^3\sum_{j=1}^{c_1}\left\{\frac{S_{1j}^{*''}(s_{1j})S_{1j}^*(s_{1j})}{S_{1j}^{*'}(s_{1j})^3}-\frac{1}{S_{1j}^{*'}(s_{1j})}\right\} \\ &= -\{U_1^{*'}(-s_0)\}^3\sum_{j=1}^{c_1}\left[\frac{S_{1j}^{*''}(s_{1j})S_{1j}^*(s_{1j})-\{S_{1j}^{*'}(U_1^*j)\}^2}{S_{1j}^{*'}(s_{1j})^3}\right]. \end{aligned}$$

Since $S_{1j}^*(s_{1j})$ is a logarithmic-convex function of s_{1j}, the term between brackets is positive. Therefore, $U_1^*(-s_0)$ is a logarithmic-convex function of s_0.

We denote by (τ,∞) the domain of $T^*(s_0)$ and by (θ_{1j},∞) the domain of $S_{1j}^*(s_{1j})$. Then $f(s_0)=T^*(s_0)U_1^*(-s_0)$ is defined on $(\tau,-\sum_{j=1}^{c_1}\theta_{1j})$. It is trivial that $s_0=0$ is a solution of the equation $f(s_0)=1$. This solution leads us to a solution $(h,s_0,s_{11},s_{12},\ldots,s_{1c_1})=(1,0,0,0,\ldots,0)$ of (3.2). Since $f(s_0)$ is a logarithmic-convex function with $\lim_{s_0\downarrow\tau}f(s_0)=\lim_{s_0\uparrow-\sum\theta_{1j}}f(s_0)=\infty$, the equation $f(s_0)=1$ has another solution $s_0=\sigma_0$. We note that, if $\rho_1<1$ then

$$1<\frac{1}{\rho_1}=T^{*'}(0)\sum_{j=1}^{c_1}\frac{1}{S_{1j}^{*'}(0)}=\frac{T^{*'}(0)}{U_1^{*'}(0)}.$$

Since $U_1^{*'}(0)$ is negative, we have $f'(0)=T^{*'}(0)-U_1^{*'}(0)<0$. Thus we know that $\sigma_0>0$. From the monotonicity of $S_{1j}^*(s_{1j})$, this solution leads us to another solution of (3.2)

$$(h,s_0,s_{11},s_{12},\ldots,s_{1c_1})=(\eta_1,\sigma_0,\sigma_{11},\sigma_{12},\ldots,\sigma_{1c_1}),$$

where $\eta_1<1$ and $\sigma_{1j}<0$. ∎

Similar argument to the proof of Lemma 8.1 can be applied to the following three lemmas.

Lemma 8.2. The system of equations (3.3) has two solutions, one of which is $(1, \sigma_0, \sigma_{11}, \ldots, \sigma_{1c_1}, 0, \ldots, 0)$.

Lemma 8.3. The system of equations (3.7) has two solutions, one of which is $(1, 0, 0, \ldots, 0)$. For the other solution $(\overline{\eta}_2, \overline{\sigma}_0, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{2c_2})$, $\overline{\eta}_2 < 1$, $\overline{\sigma}_0 > 0$ and $\overline{\sigma}_{2j} < 0$.

Lemma 8.4. The system of equations (3.8) has two solutions, one of which is $(1, \overline{\sigma}_0, 0, \ldots \ldots, 0, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{2c_2})$.

The following four lemmas are used in Sections 5 and 6. Since all of them can be proved in a similar way, here we give a proof of Lemma 8.5 only.

Lemma 8.5. If $\eta_2 < 1$, then

$$\frac{1}{\eta_1} \sum_{j=1}^{c_1} \frac{1}{S_{1j}^{*'}(\sigma_{1j})} > \sum_{j=1}^{c_2} \frac{1}{S_{2j}^{*'}(0)}. \tag{8.8}$$

Proof. Similar to the proof of Lemma 8.1, we introduce two functions

$$\begin{array}{ll} U_1^*(U_1^*) = S_{1j}^*(s_{1j}(U_1^*)), & U_1^* = s_{11} + s_{12} + \cdots + s_{1c_1}, \\ U_2^*(U_2^*) = S_{2j}^*(s_{2j}(U_2^*)), & U_2^* = s_{21} + s_{22} + \cdots + s_{2c_2}, \end{array} \tag{8.9}$$

and consider the equation

$$f_2(U_2^*) = U_1^*(-\sigma_0 - U_2^*)U_2^*(U_2^*) = \eta_1^{-1}.$$

Since $f_2(U_2^*)$ is logarithmic-convex, this equation has two roots, 0 and σ_2. If $\eta_2 < 1$, then $\sigma_2 < 0$ from (8.9) and $f_2'(0) > 0$. This implies that

$$f_2'(0) = -U_1^{*'}(-\sigma_0) + U_1^*(-\sigma_0)U_2^{*'}(0) > 0,$$

and hence

$$\frac{1}{\eta_1 U_1^{*'}(-\sigma_0)} > \frac{1}{U_2^{*'}(0)}.$$

From a similar equation to (8.6), we prove (8.8). ∎

Lemma 8.6. $\dfrac{\eta_1}{T^{*'}(\sigma_0)} < \dfrac{1}{\eta_1} \displaystyle\sum_{j=1}^{c_1} \frac{1}{S_{1j}^{*'}(\sigma_{1j})}$.

Lemma 8.7. If $\overline{\eta}_1 < \overline{\eta}_2$, then $\dfrac{1}{T^{*'}(\overline{\sigma}_0)} > \dfrac{1}{\overline{\eta}_2} \displaystyle\sum_{j=1}^{c_1} \frac{1}{S_{1j}^{*'}(0)}$.

Lemma 8.8. $\dfrac{\overline{\eta}_2}{T^{*'}(\overline{\sigma}_0)} > \dfrac{1}{\overline{\eta}_2} \displaystyle\sum_{j=1}^{c_2} \frac{1}{S_{2j}^{*'}(\overline{\sigma}_{2j})}$.

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Kou FUJIMOTO: `fujimoto@is.titech.ac.jp`
Yukio TAKAHASHI: `yukio@is.titech.ac.jp`
Naoki MAKIMOTO: `makimoto@is.titech.ac.jp`
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Ookayama, Meguro-ku, Tokyo 152 Japan